change basis: $\gamma = \alpha + \beta$. $\ker(\partial_1)$ gen by $\{\gamma, \beta\}$
$\mathrm{im}(\partial_2)$ gen by $\{-\gamma, -\gamma+2\beta\} \cong \{\gamma, 2\beta\}$

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$H_1^{\Delta}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Classification of finite abelian groups

$G \cong \mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_n}$ where q_i are (not nec distinct) prime powers.

this is unique up to order of the factors.

Note: $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}/6\mathbb{Z}$. $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}/4\mathbb{Z}$.

$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0$

H_*: $\mathbb{Z}/2\mathbb{Z}$.

$\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to 0$.

H_*: $\mathbb{Z}/3\mathbb{Z}$.

but $\to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0$

$(1,0) \longmapsto (2,0)$
$(0,1) \longmapsto (0,3)$ H_*: $\mathbb{Z}$.

small computations: ad hoc methods work.

large computations: Smith normal form. $\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \to C_{n-1} \to \cdots$

$H_n = \ker(\partial_n) / \mathrm{im}(\partial_{n+1})$ aim: make ∂_n diagonal matrix. $C_n \xrightarrow{\partial_n} C_{n-1}$.

basis: $b_1 \dots b_k$ $a_1 \dots a_\ell$.

$\partial_n = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ ~~$k\times\ell$~~ matrix $\ell \times k$

can: change basis in $C_{n-1} \leftrightarrow$ row operation
" " $C_n \leftrightarrow$ col operation

i.e. can do row/col operations, but no division!!

$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$

$X = \mathbb{RP}^2$ example: $0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^2 \to 0$.

E, D $\quad a, b, c \quad v, w$.

$D \longmapsto -a+b-c$ $\quad a \mapsto -v+w$
$E \longmapsto -a+b+c$ $\quad b \mapsto -v+w$
$c \mapsto v-v=0$

note $\ker(\mathbb{Z}^a \to \mathbb{Z}^b)$.
always free abelian gp, so just need to know the rank, and $\mathrm{im}(\partial)$.

$\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. change basis in $C_0 \leftrightarrow$ row ops. in ∂_1

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. change basis in $C_1 \leftrightarrow$ col ops in ∂_1, row ops in ∂_2

so take $\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\Rightarrow \ker(\ )$ rank 2.

and do row/col ops: change basis in $C_2 \leftrightarrow$ col ops in ∂_2

$\begin{bmatrix} +1 & 1 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$.

$\begin{bmatrix} -1 & 0 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so image is $\mathbb{Z} \oplus 2\mathbb{Z} \leq \mathbb{Z}^2$, quotient $\mathbb{Z}/2\mathbb{Z}$.

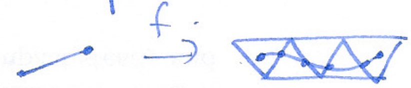
we've defined Δ-complex homology $H_n^\Delta(X)$.

Q: does $H_n^\Delta(X)$ depend on Δ-structure on X. A: no.

Q: does a map $f: X \to Y$ induce a map $f_*: H_n^\Delta(X) \to H_n^\Delta(Y)$?

A: yes, simplicial approximation theorem.

Thm If K is a finite simplicial complex, and L is an arbitrary simplicial complex, then any map $f: K \to L$ is homotopic to a map which is simplicial wrt to some iterated barycentric subdivision of K.



Fancier version of homology:

Singular homology

Defn A singular n-simplex is a cts map $\sigma: \Delta^n \to X$

Let $C_n(X)$ be the free abelian group with basis consisting of singular n-simplices. Elements are n-chains: $\sum_{i=1}^{n} n_i \sigma_i$

The boundary map is $\partial_n: C_n(X) \to C_{n-1}(X)$ given by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]$$

Remark previous argument shows that $\partial_n \partial_{n+1} = 0$ $(\partial^2 = 0)$.

Defn The singular homology groups $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$

Remarks
- homeomorphic spaces have isomorphic $H_n(X)$.
- how do we calculate $H_n(X)$? (e.g. if $X = X^{(n)}$ is $H_{n+k}(X) = 0$? $k \geq 1$?)
- 1-cycles are collections of loops, 2-cycles are maps of surfaces.

Propn If X has path components X_α, then $H_*(X) \cong \bigoplus_\alpha H_*(X_\alpha)$.

Proof $\sigma(\Delta^n)$ is path connected, so lies in a single path component. $\square$.

Propn X path connected, $\neq \emptyset$, then $H_0(X) \cong \mathbb{Z}$.

Proof $H_0(X) = C_0(X) / \operatorname{im} \partial_1$ $\qquad \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$

define a homomorphism $\epsilon: C_0(X) \to \mathbb{Z}$

$$\sum n_i \sigma_i \mapsto \sum n_i$$