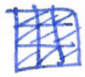
Δ-complexes vs triangulations ← each simplex is embedded in X  x  ✓

note : . each edge / subsimplex in a Δ-complex is oriented  ✓  x.

. face identifications always preserve the orderings of the vertices.

. no two distinct points in the interior of a face may be identified.

so $X = \coprod$ open simplices $e^n_\alpha \subset \Delta^n$, with $\sigma_\alpha : \Delta^n \to X$ restricts to a homeomorphism on e^n_α. Fact : this gives a CW-complex structure to X.

Simplicial Homology

X Δ-complex.

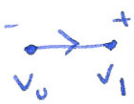
$\Delta_n(X)$ free abelian group with basis open n-simplices e^n_α of X.

↑ elements are formal sums $\sum_\alpha n_\alpha e^n_\alpha$ $\quad n_\alpha \in \mathbb{Z}$ (or $\sum_\alpha n_\alpha \sigma_\alpha$).

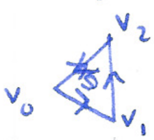
called chains.

The boundary of an n-simplex is an (n-1)-chain.

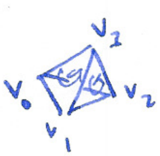
Examples • $[v_0]$ $\quad \partial[v_0] = \phi$



$$\partial[v_0, v_1] = [v_1] - [v_0]$$



$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$



$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2].$$

In general : $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

on basis elements : $\sigma_\alpha \longmapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$.

extend linearly on sums.

Lemma $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero. $\quad \partial_{n-1}\partial_n = 0$, $\partial^2 = 0$.

Proof $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]}$

$$\partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots \hat{v}_i, \dots, v_n]} + \sum_{i<j} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots \hat{v}_j, \dots v_n]}$$

$= 0 \quad \square.$

algebraic setup: $\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$

Def$^{\underline{n}}$ A chain complex is a sequence of abelian groups C_n, with homomorphisms $C_n \xrightarrow{\partial_{n-1}} C_{n-1}$ s.t. $\partial_{n-1}\partial_n = 0$.

Note: $\partial_n \partial_{n+1} = 0 \implies \mathrm{im}(\partial_{n+1}) \leq \ker(\partial_n)$.

define the n-th homology group of the chain complex to be

$$H_n = \ker(\partial_n) / \mathrm{im}(\partial_{n+1})$$

elements of $\ker(\partial_n)$ are cycles
elements of $\mathrm{im}(\partial_n)$ are boundaries

elements of H_n are cosets of $\mathrm{im}(\partial_{n+1})$ called homology classes.

two equivalent cycles c_1, c_2 are called homologous if $c_1 - c_2 \in \mathrm{im}\, \partial_n$.

set $C_n = \Delta_n(X)$, then $H_n^{\Delta}(X)$ is the n-th simplicial homology group

Examples $X = \{pt\}$

$$\left.\begin{array}{l} \cdots \to C_2 \to C_1 \to C_0 \to 0 \\ \cdots \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0 \\ \to 0 \to 0 \to \mathbb{Z} \to 0 \end{array}\right\} \quad \begin{array}{l} H_n(X) = 0 \\ \quad n > 0 \\ H_0(X) \cong \mathbb{Z}. \end{array}$$

$X = e = [v_0, v_1]$

$$\left.\begin{array}{l} \cdots \to C_2 \to C_1 \to C_0 \to 0 \\ \cdots \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0 \\ 0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0 \end{array}\right\} \quad \begin{array}{l} H_n(X) = 0 \\ \quad n \geq 2 \\ H_1(X) = 0 \\ H_0(X) \cong \mathbb{Z}. \end{array}$$

$$[v_0, v_1] \xmapsto{\partial_1} [v_1] - [v_0]$$

$\ker(\partial_1) = 0$, so $H_1^{\Delta}(X) = \ker(\partial_1)/\mathrm{im}(\partial_2) = 0/0 = 0$. $\quad H_1(X) = 0$.

$\ker(\partial_0) = \mathbb{Z}^2$, $\mathrm{im}(\partial_1) = \{n([v_1] - [v_0]) \mid n \in \mathbb{Z}\} = \{n(1,-1) \mid n \in \mathbb{Z}\} = \{(n,-n) \mid n \in \mathbb{Z}\}$

note $\mathbb{Z}^2$: standard basis $\{\underset{(1,0)}{[v_0]}, \underset{(0,1)}{[v_1]}\}$, but $\{(1,-1), (1,0)\}$ also a basis.

so $H_0^{\Delta}(X) \cong \mathbb{Z}$.

$X = S^1$ 

$$\cdots \to C_2 \to C_1 \to C_0 \to 0$$
$$\cdots \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$

$e = [v_0, v_1]$, glued to $[v]$.

$$\to 0 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0$$
$$[e]. \qquad [v].$$

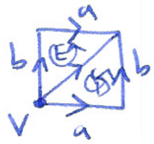
$H_n^\Delta(S^1) = 0 \quad n \geq 2.$

$[e] \overset{\partial_1}{\longmapsto} [v] - [v] = 0.$

$H_1^\Delta(S^1) = \mathbb{Z}$

$H_0^\Delta(S^1) = \mathbb{Z}.$

$$\to 0 \to \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \to 0.$$

$X = T^2 =$ 

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0.$$
$$\cdots \to \Delta_3(X) \to \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \to 0$$
$$0 \qquad \mathbb{Z}^2 \qquad \mathbb{Z}^3 \qquad \mathbb{Z}$$

bases: $[E],[D]$ $\quad [a],[b],[c].$ $\quad [v].$

$H_n^\Delta(T^2) = 0 \quad n \geq 3.$

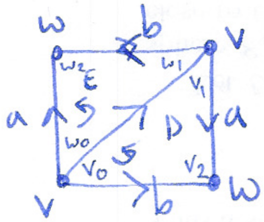
$a \longmapsto 0$
$b \longmapsto 0$
$c \longmapsto 0.$

$H_2^\Delta(T^2) = \ker(\partial_2)/\operatorname{im}\partial_3$
$= \mathbb{Z}/0 = \mathbb{Z}.$

$D \longmapsto a+b-c$
$E \longmapsto -a-b+c$

$H_1^\Delta(T^2) = \ker\partial_1/\operatorname{im}\partial_2 = \mathbb{Z}^3/\mathbb{Z}.$ $\leftarrow$ check $\{(1,1,-1),(1,0,0),(0,1,0)\}$ is a basis.
$\cong \mathbb{Z}^2.$

$H_0^\Delta(T^2) = \mathbb{Z}/0 \cong \mathbb{Z}.$

$X = \mathbb{R}P^2$ 

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0$$
$$\Delta_3(X) \to \Delta_2(X) \to \Delta_1(X) \to \Delta_0(X) \to 0$$
$$0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$
$$E, D \qquad a, b, c \qquad v, w$$

$D \longmapsto -a+b-c$
$E \longmapsto -a+b+c$
$a \longmapsto -v+w$
$b \longmapsto -v+w$
$c \longmapsto v-v=0$

$H_n^\Delta(X) = 0 \quad n \geq 3.$

$H_2^\Delta(X) = 0$ as $\ker(\partial_2) = 0$

$H_0^\Delta(X) = \mathbb{Z}$ as $\mathbb{Z}^2$ has basis $\{(-1,+1),(1,0)\}$.

$H_1^\Delta(X) = \ker(\partial_1)/\operatorname{im}(\partial_2)$

$\ker(\partial_1)$ gen by $\{a-b, c\}$
$\operatorname{im}(\partial_2)$ gen by $\{-a+b-c, -a+b+c\}.$

change basis: $\alpha = a-b$
$\beta = c$

$\ker(\partial_1)$ gen by $\{\alpha, \beta\}.$
$\operatorname{im}(\partial_2)$ gen by $\{-\alpha-\beta, -\alpha+\beta\}$