§2 Homology

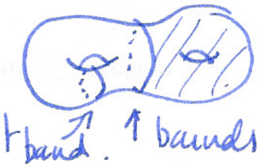
<u>motivation</u>: $\pi_1(X)$ depends only on $X^{(2)}$, want higher dim invariants.

why not $\pi_k(X, x_0)$ = homotopy classes of maps $(S^k, s_0) \to (X, x_0)$

<u>Fact</u> $\pi_k(X)$ abelian for $k \geq 2$, hard to compute

$\pi_n(S^n) \cong \mathbb{Z}$, $\pi_3(S^2) = \mathbb{Z}$, Hopf fibration:  $\mathbb{C}$-lines in $\mathbb{C}^2$.

<u>Intuition</u>: look at oriented k-dim subsets, with no boundary, which don't bound $(k+1)$-dim subsets.

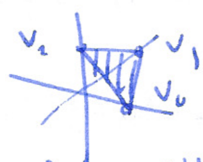


§2.1 Simplicial and singular homology

<u>Δ-complexes</u>

n-simplex: convex hull of $(n+1)$ points $v_0, \dots, v_n$ in $\mathbb{R}^m$ that do not lie in a plane of dim $< n$, equivalently, the vectors $v_0, v_1 - v_0, \dots, v_n - v_0$ are linearly independent.

notation: $[v_0, v_1, \dots, v_n]$



standard n-simplex v_i = basis vectors

barycentric coordinates $\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \}$.

we want to keep track of the order of the vertices, so $[v_0, \dots, v_n]$ is the <u>ordered</u> set of vertices. This orders every subsimplex, e.g. each edge $[v_i, v_j]$

A <u>face</u> of a simplex $[v_0, \dots, v_n]$ is any (not nec proper) (ordered) subset of $[v_0, \dots v_n]$

<u>Example</u> $[v_0, v_1, v_2]$: 3 vertices $[v_0], [v_1], [v_2]$ 0-simplices

3 edges $[v_0, v_1], [v_0, v_2], [v_1, v_2]$. 1-simplices

1 face $[v_0, v_1, v_2]$. 2-simplices.

<u>Defn</u> A <u>Δ-complex</u> is the quotient space of a collection of disjoint simplices obtained by identifying certain of their faces, by the canonical linear maps preserving the ordering of the vertices.

<u>Examples</u>