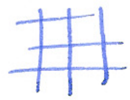
Cayley graphs / Cayley complexes

G group, presentation $\langle g_\alpha \mid r_\alpha \rangle \rightsquigarrow$ 2-d complex X_G

Cayley graph: vertices $\leftrightarrow$ group elements
edges $\leftrightarrow$ (g,h) if $g^{-1}h \in$ generating set.

Cayley complex: glue ~~on~~ a 2-cell for each r_α, starting at every vertex.

Example $\langle a,b \mid ab=ba \rangle$ 

Fact Cayley complex $= \tilde{X}_G$ universal cover of X_G.

Graphs and free groups

graph: edges
vertices 

Prop^n $\pi_1(X,x_0)$ is free if X is a graph, generating set $[f_\alpha]$ for each edge α in $X \setminus T$, where T is a maximal tree. $\square$ (van-Kampen).

Lemma $\tilde{X} \to X$ covering space, then X graph $\Rightarrow \tilde{X}$ graph.

Proof — edge has pre-image $\equiv$ collection of edges.
$\times$ vertex $\times$ $\times$ $\times$ collection of vertices.

Thm Every subgroup of a free group is free.

Proof subgroup $\leftrightarrow$ cover $=$ graph, so π_1 free $\square$.

$K(G,1)$'s / Eilenberg-Maclane spaces.

Def^n If $\pi_1(X,x_0)=G$, and $\tilde{X}$ universal cover is contractible, then we say X is a $K(G,1)$ space.

Examples: S^1, graphs, surfaces, $\mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z},1)$, products of these.

Prop^n Every group has a $K(G,1)$

Proof EG Δ-complex n-simplices $[g_0,g_1,\dots,g_n]$ elements of G

$\partial(n\text{-simplex}) = \bigcup_i [g_0,\dots\hat{g}_i,\dots,g_n]$.

<u>claim</u>: EG contractible: do linear homotopy for $x \in [g_0, \dots, g_n]$ to $[e]$ in $[e, g_0, \dots, g_n]$. This is well defined, as if we restrict to each face, i.e. if $x \in [g_0, \dots, \hat{g}_i, \dots g_n]$ then get linear homotopy in $[e, g_0, \dots, \hat{g}_i, \dots, g_n]$.

(<u>note</u>: not deformation retraction as $[e]$ goes to $[e]$ along $[e, e]$.

$G \curvearrowright EG$ by $g[g_0, \dots, g_n] = [gg_0, \dots, gg_n]$

<u>claim</u>: action is a covering space action

so the quotient map $EG \to EG/G$ is the universal cover of $BG = EG/G$, and BG is a $K(G,1)$ $\square$.

<u>Remark</u> BG has a single vertex $[e]$, but is infinite dimensional.

<u>Fact</u> If G has torsion, then $K(G,1)$ is infinite dimensional.

<u>Thm</u> The homotopy type of a CW-complex $K(G,1)$ is uniquely determined by G. $\square$

<u>Graphs of groups</u> Γ graph, connected

vertex $v \leftrightarrow$ vertex group G_v
edge $e = (v, w) \leftrightarrow$ edge group G_e and homomorphisms
$\varphi_v : G_e \to G_v$
$\varphi_w : G_e \to G_w$.

topological construction:

vertex $v \leftrightarrow K(G_v, 1)$. X_v

edge $e = (v, w) \leftrightarrow X_e$ $K(G_e, 1)$, w/ gluing maps

$X = \coprod X_v, X_e$ / gluings. $\quad \varphi_v : X_e \to X_v$, $\varphi_w : X_e \to X_w$ inducing φ_v, φ_w $\quad$ group is $\pi_1(X)$.

<u>Example</u>. $G \overset{\bullet\!-\!-\!\bullet}{\underset{\{1\}}{}} H = G * H$.

$G \bigcirc G$ $\varphi_v = \varphi_v^{-1}$ mapping tori.

<u>Thm</u> $G = \pi_1(X)$ acts on tree obtained from $\tilde{X}$.