Prop^n X path connected, locally path connected, then two covering spaces $p_1:\tilde{X}_1\to X$, $p_2:\hat{X}_2\to X$ are isomorphic by $f:\tilde{X}_1\to\tilde{X}_2$ taking the basepoint $\tilde{x}_1\in p_1^{-1}(x_0)$ to $\tilde{x}_2\in p^{-1}(x_0)$ iff $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\hat{X}_2,\tilde{x}_2))$ (equal, not isomorphic, conjugate).

Proof ($\Rightarrow$) Let $f:(\tilde{X}_1,\tilde{x}_1)\to(\tilde{X}_2,\tilde{x}_2)$ be an isomorphism. Then $p_1=p_2f$ and $p_2=p_1f^{-1}$ $\Rightarrow$ $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2,\tilde{x}_2))$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[r, "f"] \arrow[dr, "p_1"'] & \tilde{X}_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

($\Leftarrow$) suppose $p_{1*}(\pi_1(\tilde{X}_1,\tilde{x}_1)) = p_{2*}(\pi_1(\hat{X}_2,\tilde{x}_2))$

lifting criterion gives $\tilde{p}_1:\hat{X}_1\to\tilde{X}_2$, $\tilde{p}_2:\tilde{X}_2\to\tilde{X}_1$ consider

$$\begin{tikzcd} \tilde{X}_1 \arrow[r, "\tilde{p}_1"] \arrow[dr, "p_1"'] & \hat{X}_2 \arrow[r, "\hat{p}_2"] \arrow[d, "p_2"] & \tilde{X}_1 \arrow[dl, "p_1"] \\ & X & \end{tikzcd}$$

but $\tilde{p}_2\tilde{p}_1$ is a lift of p_1, but $id_{\tilde{X}_1}$ is also a lift of p_1, so uniqueness of lifts $\Rightarrow$ $\tilde{p}_2\tilde{p}_1 = id_{\tilde{X}_1}$ similarly $\tilde{p}_1\tilde{p}_2 = id_{\tilde{X}_2}$, $\Rightarrow$ both $\tilde{p}_1,\tilde{p}_2$ homeos. $\square$.

Final bit: change of basepoint $\longleftrightarrow$ conjugation in $\pi_1(X,x_0)$.

$\Rightarrow$

$$\begin{tikzcd} \tilde{X},\tilde{x}_1 \arrow[d] & \cdot\tilde{x}_1 \arrow[d, bend right, "h"'] & \\ & \cdot\tilde{x}_2 & p^{-1}(x) \\ X,x_0 & \cdot x_0 & \end{tikzcd}$$

recall: change of basepoint map $\beta_h:\pi_1(\tilde{X},\tilde{x}_1)\to\pi_1(\tilde{X},\tilde{x}_2)$.

image under p_*: $hf\bar{h}\mapsto[h][f][h]^{-1}$

as p_*h is a loop in $\pi_1(X,x_0)$.

$\Leftarrow$ for each conjugate gHg^{-1} choose a lift $\tilde{g}$ of g and define $\beta_{[\tilde{g}]}$ as above, to produce cover $(\tilde{X},\tilde{g}(1))$ with $p_*(\pi_1(\tilde{X},\tilde{g}(1))) = gHg^{-1}$, but then uniqueness of covers means $\tilde{X}_{gHg^{-1}} = \tilde{X},\tilde{g}(1)$. $\square$.

Remark partial order on covers by covering $\longleftrightarrow$ partial order on subgroups by inclusion.

Representing covering spaces by permutations

$S_n = \mathrm{Sym}(n) =$ permutation of $\{1, 2, \ldots n\}$

$\mathrm{Sym}(X) =$ permutation of X (discrete topology).

An *action* of a group G on a discrete set X is a homomorphism $\rho: G \to \mathrm{Sym}(X)$.

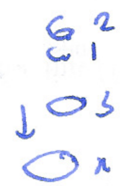
(disconnected) covering spaces $\leftrightarrow$ permutation actions

connected covering spaces $\leftrightarrow$ transitive permutation actions.

the set is always ~~X~~ $p^{-1}(x_0)$.

Example ①  $\pi_1(X, x_0) \cong \mathbb{Z} \curvearrowright \{1,2,3\}$.

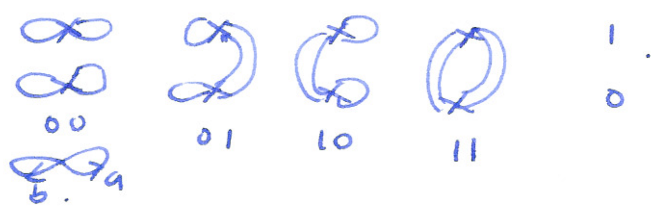
$1 \mapsto (123)$

 $1 \mapsto (12)(3)$.

② 2-fold covers of $S^1 \vee S^1$. $\rho: F_2 \to \mathrm{Sym}(\{1,2\}) \cong \mathbb{Z}_2$.

$\langle a, b | \rangle \to \mathbb{Z}_2$. $\quad a \mapsto 0,1$
$\quad\quad 0,1. \quad\quad b \mapsto 0,1$.

$0 = (1)(2)$.
$1 = (12)$



③ 3-fold cover of $S^1 \vee S^1$ $\quad \rho: F_2 \to \mathrm{Sym}(\{1,2,3\}) \leftarrow$ order 6 non-abelian.

permutations up to conjugacy: $(1)(2)(3)$. $(12)(3)$. (123)

possibilities:

a	$(1)(2)(3)$	$(1)(2)(3)$.	$(1)(2)(3)$	$(12)(3)$	
b	$(1)(2)(3)$	$(12)(3)$	(123)	$(1)(2)(3)$	etc. ...

$\underbrace{\quad\quad}_{\text{not transitive}}$ $\quad \underbrace{\quad}_{\text{transitive}}$.

In general: A cover $p: \tilde{X} \to X$ gives a permutation rep. of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ by $[g] \mapsto \tilde{g}(1)$, well defined by homotopy lifting property.

Converse reconstruct $\tilde{X}$ from action of ~~X~~ $\pi_1(X, x_0)$ on $p^{-1}(x_0) = F$.

let $\tilde{X}_1$ be the universal cover, and define $h: \tilde{X}_1 \times F \to \tilde{X}$

$$([\gamma], \tilde{x}_0) \mapsto \tilde{\gamma}_x(1)$$

where $\tilde{\gamma}_x(1)$ is the lift of γ starting at $\tilde{x} \in F$.

note h is continuous as open sets in $\tilde{X}_1 \times F$ look like $U_{[\gamma]} \times \{\tilde{x}\}$ so in fact local homeomorphism.

h is surjective as X path connected.

so can take quotient $\tilde{X}_1 \times F/\sim$ where $([\gamma], \tilde{x}) \sim ([\gamma'], \tilde{x}')$ if
$h(\quad '' \quad) = h(\quad '' \quad)$.

suppose $h([\gamma], \tilde{x}) = h([\gamma'], \tilde{x}')$, then:

so $\tilde{x}_0' = L_{\gamma\bar{\gamma}'}(\tilde{x}_0)$

let $\lambda =$ loop $\gamma\bar{\gamma}$ in X, then $h([\gamma], \tilde{x}_0) = ([\lambda\gamma], L_\lambda(\tilde{x}_0))$

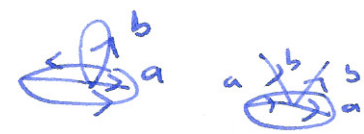
and this works for any loop λ, so we get a map $\tilde{X}_1 \times F/\sim \;\to \tilde{X}$,

h local homeo, so $\tilde{X}$ depends only on L (homotopy class of L)

Corollary n-sheeted covers of $X \leftrightarrow$ reps $\pi_1(\tilde{X}, x) \to \mathrm{Sym}(n)$ (up to conjugacy).

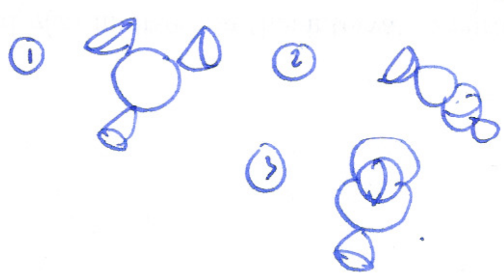
Example construct all (connected) 3-fold covers of $\mathbb{RP}^2 \vee S^1$

$\pi_1(\mathbb{RP}^2 \vee S^1) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} = \langle a, b \mid a^2 \rangle \to \mathrm{Sym}(3)$



$a \mapsto (1)(2)(3)$, $b \mapsto (123)$ ① (only transitive choice)

or $a \mapsto (12)(3)$, $b \mapsto (1)(23)$ ②; $b \mapsto (123)$ ③



Deck transformations and group actions

$p: \tilde{X} \to X$ covering space. The covering space isomorphisms $\tilde{X} \to \tilde{X}$ are called the deck transformations or covering space transformations of the cover. These form a group $G(\tilde{X})$ under composition.

Example $\mathbb{R} \xrightarrow{} S^1$, deck transformation $x \mapsto x+1$, $G(\mathbb{R} \to S^1) \cong \mathbb{Z}$.

$S^1 \xrightarrow{p} S^1$, $z \mapsto z^n$, $G(z \mapsto z^n) \cong \mathbb{Z}/n\mathbb{Z}$.

By the unique lifting property a deck transformation is determined by where it sends $\tilde{x}_0$ basepoint in $\tilde{X}$, so if it fixes a point, it is the identity.

<u>Defn</u> A covering space is <u>normal</u> or <u>regular</u> if for every $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$, there is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_0'$.

<u>Propn</u> Let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space, and let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Then a) $\tilde{X} \to X$ is <u>normal/regular</u> iff $H \subseteq \pi_1(X, x_0)$ is a normal subgroup of $\pi_1(X)$

b) the deck transformation group $G(\tilde{X}) \cong N(H)/H$ where $N(H)$ is the <u>normalizer</u> of H, i.e. the largest subgp in which H is normal

In particular, if $\tilde{X}$ is regular, $G(\tilde{X}) = \pi_1(X, x)/H$.

if $\tilde{X}$ is universal cover $G(\tilde{X}) = \pi_1(X, x_0)$.

<u>Prof</u> recall: change of basepoint $\tilde{x}_0$ to $\tilde{x}_1$ in $p^{-1}(x_0)$, corresponds to conjugation of H in $\pi_1(X, x_0)$ by $[\gamma]$, where $\tilde{\gamma}_{x_0}(1) = \tilde{x}_1$.

so $[\gamma] \in N(H) \iff$

$$\underset{\substack{\| \\ p_*(\pi_1(\tilde{X}, \tilde{x}_1))}}{[\gamma] H [\gamma]^{-1}} = \underset{\substack{\| \\ p_*(\pi_1(\tilde{X}, \tilde{x}_0))}}{H}.$$

lifting criterion $\Rightarrow$ $\exists$ deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[rr, shift left, "\tilde{p}_0"] \arrow[dr, "p_0"'] & & (\tilde{X}, \tilde{x}_1) \arrow[ll, shift left, "\tilde{p}_1"] \arrow[dl, "p_1"] \\
& X &
\end{tikzcd}$$

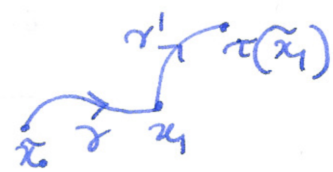
i.e. $[\gamma] \in N(H) \iff \exists$ deck transformation $\tau: \tilde{x}_0 \to \tilde{x}_1$

i.e. $\tilde{X} \to X$ regular iff $N(H) = \pi_1(X, x_0)$.

quotient: define $\phi: N(H) \to G(\tilde{X})$

$[\gamma] \longmapsto \tau_{[\gamma]}: \tilde{x}_0 \longmapsto \tilde{x}_1$

↑deck transformation



- ϕ is a homomorphism $\phi'\phi = \tau_{[\gamma]'}\tau_{[\gamma]} = \tau_{[\gamma\gamma']}$
- ϕ is surjective by above.
- ϕ is injective: kernel = loops in X lifting to loops in $\tilde{X}$

$= p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$, so $G(\tilde{X}) = N(H)/H$. $\square$.