• $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$

If $\gamma' = \gamma\eta$ then $U_{[\gamma']} = \{[\gamma\cdot\eta\cdot\mu] \mid \mu$ is a path in U with $\mu(0) = \gamma\cdot\eta(1)\} \subset U_{[\gamma]}$.

$U_{[\gamma]} = \{[\gamma\cdot\eta\bar\eta\mu] \mid \mu$ is a path in U w/ $\mu(0) = \gamma(1)\} \subset U_{[\gamma']}$.

<u>claim</u> • $U_{[\gamma]}$ form a basis for a topology on $\tilde X$.

check intersections: let $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']} = U_{[\gamma'']} \cap V_{[\gamma'']}$

$\mathcal{U}$ basis for X, so choose $W \in \mathcal{U}$ with $\gamma''(1) \in W \subset U \cap V$

then $[\gamma''] \in W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$

• $p: U_{[\gamma]} \to U$ is a homeomorphism

$$\begin{array}{ccc} & & U \\ & & \cup \\ V_{[\gamma']} & \leftrightarrow & V \\ & & \text{open} \end{array}$$

$\gamma' = \gamma\eta$.

for $[\gamma'] = [\gamma\eta]$, η path in U.

• $p: \tilde X \to X$ continuous ✓

covering space: $p^{-1}(U) = \bigsqcup_{[\gamma] \in \pi_1 X} U_{[\gamma]}$ as if $U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma]} = U_{[\gamma']}$.

Finally: $\tilde X$ simply connected

Let $[\gamma] \in \tilde X$, and let γ_t be the path along γ from $\gamma(0)$ to $\gamma(t)$.

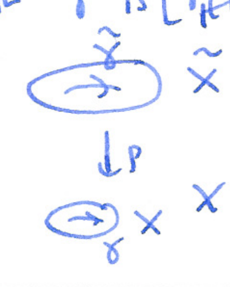
then γ_t is a path from $\tilde x = [\text{const } x_0]$ to $[\gamma]$, so $\tilde X$ path connected.

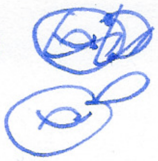
consider $p_*(\pi_1(\tilde X, \tilde x))$ = loops in X that lift to loops in $\tilde X$

a loop in $\tilde X$ starts at $[x_0]$ and ends at $[x_0]$, endpoint of path $\tilde\gamma$ is $[\gamma_1(t)]$.

so $[\gamma_1] = [x_0]$ but γ_t homotopy in $\tilde X$ from $\gamma_0 = [x_0]$ to $\tilde\gamma_1$,

so $[\gamma_1] = [x_0]$, as required. □



<u>Examples</u> ~~$T^2 \vee (ae)$~~   etc.

$T^2 \vee S^1$

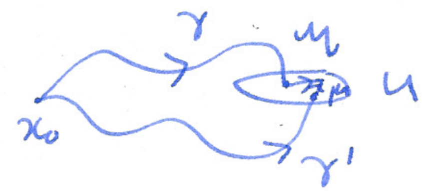
<u>Prop$^{\underline{n}}$</u> X path connected, locally path connected, semilocally simply connected then for every subgroup $H \subset \pi_1(X, x_0)$, there is a covering space $p: \tilde{X}_H \to X$ s.t. $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

<u>Proof</u> construct a quotient of the universal cover $\tilde{X}$: let $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma\bar{\gamma}'] \in H$. This is an equivalence relation: reflexive ✓ symmetric ✓ transitive: H closed under multiplication $[\gamma] \sim [\gamma']$ and $[\gamma'] \sim [\gamma'']$ then $[\gamma\bar{\gamma}'] \in H$ $[\gamma'\bar{\gamma}''] \in H$, so $[\gamma\bar{\gamma}'\gamma'\bar{\gamma}''] = [\gamma\bar{\gamma}''] \in H$. Set $\tilde{X}_H = \tilde{X}/\sim$ <u>claim</u> $\tilde{X}_H$ is a covering space.

$p: \tilde{X}_H \to X$ Let $U \subset X$, consider $p^{-1}(U) = \coprod_{\gamma \in \pi_1(X)} U_{[\gamma]} / \sim$

$[\gamma] \mapsto \gamma(1)$

If $[\gamma\eta] \in U_{[\gamma]}$ identified with $[\gamma'\mu] \in U_{[\gamma']}$



then $[\gamma\eta\bar{\mu}\bar{\gamma}'] \in H$, but now for any $[\gamma\nu] \in U_{[\gamma]}$,

$[\gamma\nu\bar{\nu}\eta\bar{\mu}\bar{\gamma}'] \in H$ so $\underset{\in U_{[\gamma]}}{[\gamma\nu]}\underset{\in U_{[\gamma']}}{[\gamma'\mu\bar{\eta}\nu]^{-1}} \in H$

so $U_{[\gamma]} = U_{[\gamma']}$. $\square$ claim.

<u>claim</u> $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

↑ loops in X which lift to loops in $\tilde{X}_H$.

γ loop in X, then $\tilde{\gamma}_t = [\gamma_t]$, so if γ lifts to a loop then $[\gamma_1] \sim [x_0]$ constant path, i.e. $[\gamma_1 \cdot x_0] \in H \Rightarrow [\gamma_1] \in H$, as required $\square$.

<u>Recall</u>: two covering spaces are isomorphic if there is a homeo $f: \tilde{X}_1 \to \tilde{X}_2$ s.t. $p_1 = p_2 f$.