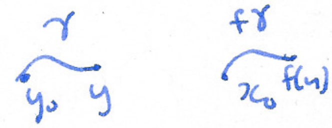
$\Leftarrow$ assume $f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(\tilde X,\tilde x_0))$

$$\begin{array}{ccc} & & \tilde X \\ & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

Let $y \in Y$ and let γ be a path from y_0 to y in Y. There is a unique lift ~~of~~ $\widetilde{f\gamma}$ of γ starting at $\tilde x_0$.

define $\tilde f(y) = \widetilde{f\gamma}(1)$

check well defined: suppose γ' is some other path from y_0 to y. Then $h_0 = f\gamma' \cdot f\bar\gamma$ is a loop in X based at x_0 with $[h_0] \in f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(\tilde X,\tilde x_0))$



so there is a homotopy h_t of h_0 to a loop h_1, that lifts to a loop $\tilde h_1$ in $\tilde X$ based at $\tilde x_0$. Apply covering homotopy property to lift this homotopy to $\tilde h_t$, $\tilde h_1$ loop at $\tilde x_0$, so $\tilde h_1$ is a loop at $\tilde x_0$. By uniqueness of lifted paths, the first half of $\tilde h_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{f\gamma}$ backwards, with common endpoint $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$, so $\tilde f$ is well defined.

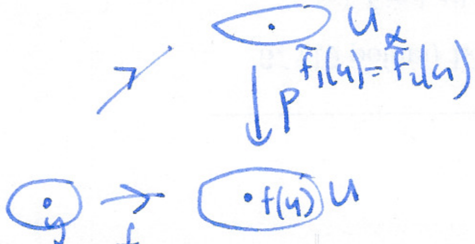
• continuous Let U be an open nbd of $f(y)$ s.t. $p|_{\tilde U} : \tilde U \to U$ is a homeomorphism, and let $V \subset U$ be a path connected nbd of $f(y)$.

For $y' \in V$, can take fixed path γ to $f(y)$, and then path η in V from $f(y)$ to y', but then η lifts by p^{-1}, so $\tilde f_g : V \to \tilde V$ by $\tilde f_g = p^{-1}$, so cts.

Unique lifting property

Prop: $p : \tilde X \to X$ covering space, $f : Y \to X$ w/ two lifts $\tilde f_1, \tilde f_2 : Y \to \tilde X$ s.t. there is a point $y \in Y$ with $\tilde f_1(y) = \tilde f_2(y)$. Then $\tilde f_1 = \tilde f_2$ on all of Y.

Proof Let U be an open nbd of y s.t. $p^{-1}(U)$ is a disjoint union of sets U_α each homeomorphic to U by $p|_{U_\alpha}$. Then $\tilde f_1(y) = \tilde f_2(y)$ implies:



so $p\tilde f_1 = p\tilde f_2 \Rightarrow \tilde f_1 = \tilde f_2$ on $f^{-1}(U) = \tilde U_\alpha$

now extend over open cover $\{U_\beta\}$ of X $\square$.

<u>Example</u> fiber bundle over S^1: $X \times I/\sim$ $\quad \phi: X \to X \quad (x,1) \sim (\phi(x), 0)$

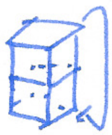
$\phi: T^2 \to T^2 \quad \phi \leftrightarrow SL_2\mathbb{Z}$.

$x \longmapsto Ax$.

$M_\phi = T^2 \times I/\sim$

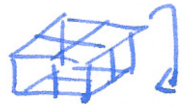
ϕ.

cyclic covers:



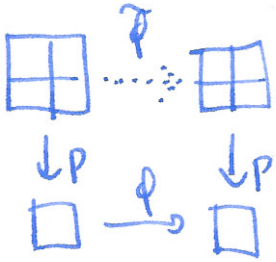
ϕ^2 etc.

Q: what about



?

need



$$\begin{tikzcd} \mathbb{Z}^2 \arrow[r, dashed, "\tilde{A}?"] \arrow[d, "p_*=2I"'] \arrow[dr, "2A"] & \mathbb{Z}^2 \arrow[d, "2I=p_*"] \\ \mathbb{Z}^2 \arrow[r, "A=\phi_*"'] & \mathbb{Z}^2 \end{tikzcd}$$

$\tilde{A}$ exists if $2A(\mathbb{Z}^2) \subset 2(\mathbb{Z}^2)$

gives even vector

even vectors

✓.

<u>Classification of covering spaces</u>

X path connected, locally path connected, semi-locally simply connected.

<u>Defn</u> X is <u>semilocally simply connected</u> if each $x \in X$ has a nbd U s.t.

$\pi_1(U,x) \xrightarrow{i_*} \pi_1(X,x)$ is trivial.

<u>not</u> semilocally simple connected: Hawaiian earring

<u>Remark</u> locally simply connected $\Rightarrow$ semilocally simply connected

locally contractible $\Rightarrow$ locally simply connected.

so CW-complexes are locally contractible.

<u>Thm</u> Let X be path connected, locally path connected, semilocally simply connected. Then there is a bijection between the set of basepoint preserving isomorphism classes of covering spaces $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. Bijection given by $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0) \leftrightarrow p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

If we ignore basepoints, this gives a correspondence between isomorphism classes of path connected covering spaces $p: \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

<u>Defn</u> Two covering spaces $p_1: \tilde{X}_1 \to X$, $p_2: \tilde{X}_2 \to X$ are isomorphic if there is a homeomorphism $f: \tilde{X}_1 \to \tilde{X}_2$ st. $p_1 = p_2 f$ ⊛

<u>Exercise</u>: this gives an equivalence relation on covering spaces.