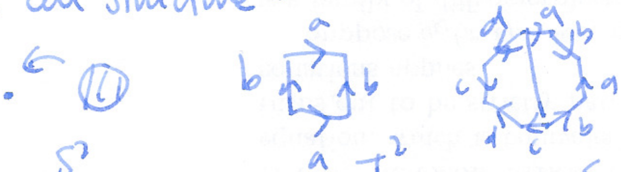
Fact Classification of closed orientable surfaces

All of the form:

S^2 $\quad T^2$ $\quad S_2$ $\quad S_3$ $\quad \dots$

with cell structure:



S^2 $\quad T^2$ $\quad S_2$

non-orientable surfaces: everything is $\#_k \mathbb{R}P^2$.

w/ boundary: just cut holes (deformation retracts to graphs).

← homotopy equivalent but not homeomorphic!

$\pi_1(S^2) = 1 \qquad \pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \qquad \pi_1(S_g) = \langle a_i, b_i \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$

Corollary these surfaces all genuinely different: $ab(\pi_1(S_g)) \cong \mathbb{Z}^{2g}$.

§1.3 Covering spaces

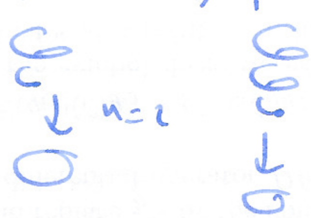
Example

$\mathbb{R} \xrightarrow{p} S^1$

property(*): there is an open cover U_α of S^1 s.t. $p^{-1}(U_\alpha)$ is homeomorphic to a discrete set $D \times U_\alpha$.

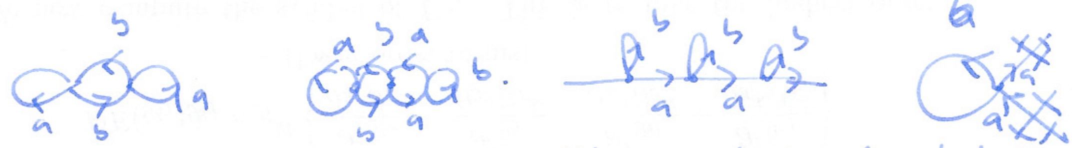
Defn A covering space $\tilde{X}$ of X, is a space $\tilde{X}$, and a map $p: \tilde{X} \to X$ s.t. there is an open cover U_α of X s.t. $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each mapped homeomorphically to U_α by p.

Examples
- $p: S^1 \to S^1$, $z \mapsto z^n$

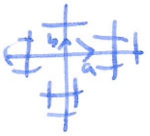


$n=2 \qquad n=3.$

- $X =$



basically any 4-valent graph with oriented edges labelled a, b s.t. every vertex has labels



Remark a covering space $\tilde{X}$ determines a subgroup $p_*(\pi_1 \tilde{X}) < \pi_1 X$.

Fact: the map p_* is injective (proof later), and there is a 1-1 correspondence between based covering spaces $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and subgp of $\pi_1(X, x_0)$.

Lifting properties

$$\begin{tikzcd} & \tilde X \arrow[d, "p"] \\ Y \arrow[ur, "\tilde f"] \arrow[r, "f"'] & X \end{tikzcd}$$

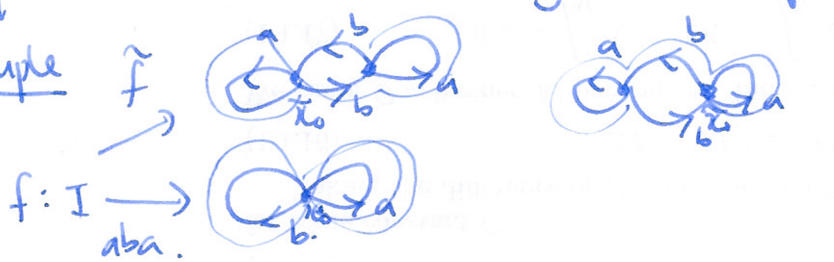
a lift of a map $f: Y \to X$ is a map $\tilde f: Y \to \tilde X$ s.t. $p\tilde f = f$.

Prop (Homotopy lifting property)
Given a covering space $p: \tilde X \to X$, a homotopy $f_t: Y \to X$ and a map $\tilde f_0: Y \to \tilde X$ lifting f_0, there is a unique homotopy $\tilde f_t: Y \to \tilde X$ lifting f_t.

Proof we proved this for $p: \mathbb{R} \to S^1$, only using property $(*)$ $\square$.

Corollary (Path lifting property) Any path $f: I \to X$ starting at x_0 has a unique lift $\tilde f: I \to \tilde X$ starting at $\tilde x_0 \in p^{-1}(x_0)$. $\square$.

Example



Remarks: · loops may lift to paths
· constant loop lifts to constant loop.

Prop Let $p: \tilde X \to X$ be a covering space, the map $p_*: \pi_1(\tilde X, \tilde x_0) \to \pi_1(X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde X, \tilde x_0)) < \pi_1(X, x_0)$ consists of (homotopy classes) of loops at x_0 which lift to loops at $\tilde x_0$.

Proof Let $\tilde f: I \to \tilde X$ be an element of the kernel of p_*, i.e. $p\tilde f$ is homotopic to the constant loop $f_1: I \to x_0$ in X, by f_t say, but there is a lift $\tilde f_t: I \to \tilde X$, starting at $\tilde f_1$ and ending at the constant map $\Rightarrow [\tilde f] = 1$ in $\pi_1(\tilde X, \tilde x_0) \Rightarrow p_*$ injective.

If $f: I \to X$ is a loop ~~at~~ with lift $\tilde f: I \to \tilde X$ which is a loop in $\tilde X$ at $\tilde x_0$, then $[f] \in p_*(\pi_1(\tilde X, \tilde x_0))$. Conversely, suppose $f': I \to X$ is a loop homotopic by f_t to a loop f having such a lift $\tilde f$, then the homotopy lifts to $\tilde f_t$ so $\tilde f_1 = \tilde f'_0$ is a lift of f', which is a loop, as $\tilde f_t(1)$ is a map $\tilde f_t(1): I \to p^{-1}(x_0) \leftarrow$ discrete set, so this map is constant, so $\tilde f'(1) = \tilde f(1) = \tilde x_0$. $\square$.

Remark $p: \tilde X \to X$ covering space. Then $|p^{-1}(x)|_{(x \in X)}$ locally constant, so if X connected, then $|p^{-1}(x)|$ constant. This number is the <u>degree</u> or <u>number of sheets</u> of the cover.