Cell complexes n-cell : closed n-ball D^n

① start with a discrete set X^0, the 0-cells (points)

② form the X^{n+1} skeleton by attaching $(n+1)$-cells to X^n by gluing maps defined on their boundaries.

Formally: chose $(n+1)$-cells D^{n+1}_α and gluing/attaching maps $\varphi_\alpha : \underset{S^{n+1}}{\overset{\cong}{\partial D^{n+1}_\alpha}} \to X^n$

then form the quotient space $X^n \coprod_\alpha D^{n+1}_\alpha / \sim$ where $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^{n+1}_\alpha$.

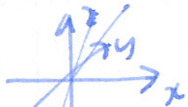
③ stop at stage X^n : finite dimensional cell complexes / CW-complex, or consider $X = \bigcup_n X^n$ $\leftarrow$ give this the weak topology: $A \subset X$ open if $A \cap X^n$ open for each n.

Examples • graphs X^0 vertices, X^1 edges 

1-cell is an edge $e_i \cong I \cong [0,1]$, $\partial e_i = \{0,1\}$ map to X^0

• S^n : $X^0 = \{pt\}$ $X^n = \underset{\text{0-cell}}{e_0} \cup \underset{n\text{-cell}}{e^n}$ attaching map $\partial e_n \xrightarrow{\varphi} pt$

(equivalent to $S^n = D^n / \partial D^n$) S^1:  S^2 , etc...

• $\mathbb{RP}^n$ space of lines in $\mathbb{R}^{n+1}$  topology: $\mathbb{R}^{n+1} / \{0\} / \sim$ $v \sim \lambda v$ $\lambda \in \mathbb{R} \setminus \{0\}$.

equivalently, $\mathbb{RP}^n = S^n / \sim$ $x \sim -x$, ~~so $\mathbb{R}$~~ in fact $\mathbb{RP}^n = D^n / \sim$ $\sim$ identifies antipodal points on ∂D^n. note: $\partial D^n / \sim$ antipodal map $= \mathbb{RP}^{n-1}$, so (inductively) $\mathbb{RP}^n$ made from attaching an n-cell of $\mathbb{RP}^{n-1}$, so $\mathbb{RP}^n$ has a cell structure w/ one cell in each dimension.

$\mathbb{RP}^0$ • $\mathbb{RP}^1$  $\cong S^1$ $\mathbb{RP}^2$  $z \mapsto z^2$ (or antipodal map). etc. (or construct cell structure on S^n, equivariant wrt antipodal map)

van Kampen • $\pi_1(\text{graph}) =$ free group.

• $\pi_1(S^n)$ $(n \geq 2) = 1$ • union of two discs w/ intersection S^{n-1}. 

Facts · $\pi_1(X)$ depends only on $\pi_1(X^{(2)})$ · $X^{(2)} \rightsquigarrow$ gives presentation for $\pi_1 X$.

Prop^n X path connected, $Y = X \cup e^2_\alpha$ $\quad \varphi_\alpha : \partial e^2_\alpha \to X$ gluing map

choose path γ_α from basepoint $x_0 \in X$ to $\varphi_\alpha(s_0)$, $s_0 \in S^1$ basepoint.

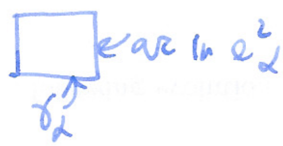
let N = normal subgroup generated by $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$

$= \langle \gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha \rangle$

then $\pi_1(Y, x_0) = \pi_1(X, x_0)/N$.

Proof (van Kampen) Replace Y by space Z w/ each path "thickened up" to $\gamma_\alpha \times I$, Z deformation retracts to Y

i.e. attach rectangle $R_\alpha = I \times I$

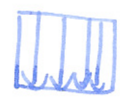


$I \times \{0\}$ attached to γ_α

$\{0\} \times I$ all identified over all α. ← choose basepoint x_0 as $\{0\} \times \{\frac{1}{2}\}$.

$\{1\} \times I$ attached to an arc in e^2_α.

top edge not attached to anything, so deformation retracts:



set $A = Z \setminus \bigcup_\alpha t_\alpha$ $\qquad t_\alpha$ basepoint in e^2_α

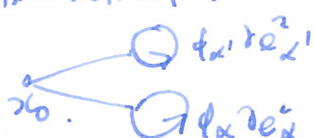
$B = Z \setminus X$ ← contractible!

van Kampen: $\pi_1 Y \cong \pi_1(A) * \pi_1(B)/N \cong \pi_1(A)/N$

where N = normal subgp generated by image of $\pi_1(A \cap B) \xrightarrow{i_*} \pi_1(A)$

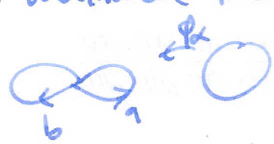
$$\begin{array}{ccc} A\cap B \hookrightarrow & A & \hookrightarrow Z \\ \hookrightarrow & B & \hookrightarrow \end{array} \qquad \begin{array}{ccc} \pi_1(A \cap B) \to & \pi_1 A & \to \pi_1 Z \\ \to & \pi_1 B & \to \end{array}$$

· $A \cap B$ deformation retracts to a graph



where $\bigcup \gamma_\alpha$ = maximal tree and each ∂e^2_α is an edge not in maximal tree, so $\pi_1(A \cap B)$ is free on $\langle \gamma_\alpha t_\alpha \bar{\gamma}_\alpha \rangle$. □.

Example



$\varphi_\alpha = ab$. $A \cap B =$

Corollary For every group G there is a 2-d cell complex X_G with $\pi_1(X_G) = G$.

Proof Choose a presentation $G = \langle g_\alpha \mid r_\beta \rangle$ [Corollary: can't classify 2-complexes, 4-manifolds]

$X^{(0)} = \{0\}$ $\qquad X^{(1)} = \{x_0\} \cup e^1_\alpha$ ← one edge for each generator.

$X^{(2)} = X^{(1)} \cup_{\varphi_\alpha} e^2_\beta$ ← one two cell for each relator, $\varphi_\beta : S^1 = \partial e^2_\alpha$ to path r_β □.