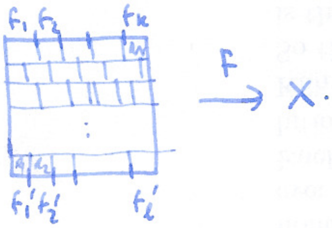
suffices to show: any two factorizations of $[f]$ are equivalent by 1, 2, as this implies $Q \xrightarrow{\Phi} \pi_1(X, x_0)$ is injective, so $Q \cong \pi_1(X, x_0)$.

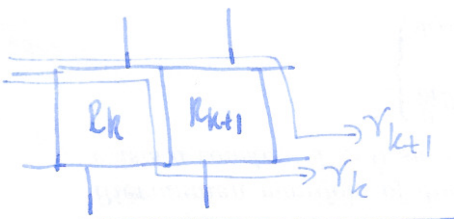
Let $[f_1][f_2]\cdots[f_k]$ and $[f_1'][f_2']\cdots[f_\ell']$ be two factorizations of $[f]$, so they are homotopic (in X!). Let $F: I\times I \to X$ be such a homotopy.



F cts, $I\times I$ compact, so we can partition F into finitely many rectangles, s.t. image of each rectangle lies in a single A_α. wlog: can divide t direction into strips, and then each strip into rectangles s.t. at most 3 rectangles share a common vertex.

number the rectangles $R_1, R_2, \ldots, R_N$ note: any path from left to right is a loop in (X, x_0)

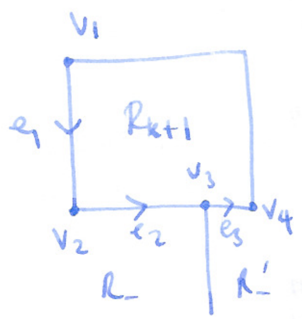
Let γ_r be the path separating $R_1, \ldots, R_r$ from $R_{r+1}, \ldots, R_N$, all γ_r homotopic paths in (X, x_0).



each vertex v lies in 3 squares, so $F(v)$ lies in $A_{\alpha_{i_1}} \cap A_{\alpha_{i_2}} \cap A_{\alpha_{i_3}}$.

---

for each vertex v choose a path g_v from x_0 to $F(v)$ s.t. $g_v \subset A_{\alpha_{i_1}} \cap A_{\alpha_{i_2}} \cap A_{\alpha_{i_3}}$. (triple intersection path connected!)



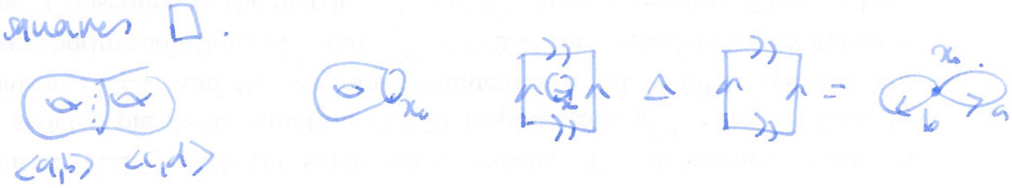
lower path γ_k: $\ldots\ e_1\ e_2\ e_3\ \ldots$

$$\simeq \ \ldots\ \underbrace{\bar{g}_{v_1} g_{v_1}\ e_1\ \bar{g}_{v_2}}_{\substack{R_k \to R_{k+1} \\ (1)}}\ \underbrace{g_{v_2}\ e_2\ \bar{g}_{v_3}}_{\substack{R_- \to R_{k+1} \\ (2)}}\ \underbrace{g_{v_3}\ e_3\ \bar{g}_{v_4}}_{\substack{R_-' \to R_{k+1} \\ (2)}}\ g_{v_4} \ldots$$

now homotope lower path to upper path across the square in $A_{\alpha_{k+1}}$. (1).

continue across all squares $\square$.

Examples $\pi_1(S_2)$



$$\pi_1(S_2) \cong \langle a,b\rangle * \langle c,d\rangle / N \qquad N = \langle [a,b] = [c,d] \rangle.$$

so $\pi_1(S_2) \cong \langle a,b,c,d \mid [a,b][c,d] \rangle$.