Thm (general version) Let X be the union of open path connected sets A_α, each containing the basepoint x_0, and $A_\alpha \cap A_\beta$ path connected, then $\Phi: \ast_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective. Furthermore, if $A_\alpha \cap A_\beta \cap A_\gamma$ path connected, then $\ker \Phi = N$ normal subgroup generated by $i_{\alpha\beta}(\omega) \cdot i_{\beta\alpha}(\omega)^{-1}$

$$\begin{tikzcd} & A_\alpha \arrow[dr, hook] & \\ A_{\alpha\cap\beta} \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, bend left=60, "i_{\alpha\beta}"] \arrow[rr, bend right=60, "i_{\beta\alpha}"'] & & X \\ & A_\beta \arrow[ur, hook] & \end{tikzcd}$$

and so $\pi_1(X, x_0) \cong \ast \pi_1(A_\alpha, x_0)/N$.

Example Wedge sum / one point union.

$X \vee Y = X \cup Y/\sim$ with $x_0 \sim y_0$ and no other points identified. (general: $\bigvee_\alpha X_\alpha$).

 $S^1 \vee S^1$ "nice" wedge sums: $x_0 \in \underset{\text{open nbd}}{U_\alpha} \subset X_\alpha$

where U_α deformation retracts to x_0

$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) \ast \pi_1(S^1)/N$, but x_0 has a contractible nbd  x_0, so N is trivial, so $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} \ast \mathbb{Z}$. In fact $\pi_1(\underbrace{S^1 \vee \ldots \vee S^1}_{k}) \cong F_k$.

Non-example $S^1 =$ union of two intervals  $A \cap B$ not path connected.

Example $S^2 =$ union of two discs  $A \cap B =$  $\cong S^1 \times (0,1) \simeq S^1$

so $\pi_1(S^2) \cong \underbrace{\pi_1(A) \ast \pi_1(B)}_{\text{trivial}}/N \cong 1$.

Example X connected graph, then $\pi_1 X$ free. $X =$  choose a maximal tree T. claim X contains all vertices of X. set ~~$A_0 = N_\epsilon(T)$ and $A_\alpha = N_\epsilon(\alpha)$~~ $A_\alpha = N_\epsilon(T \cup \text{edge } \alpha)$. for each edge α in $X \setminus T$.

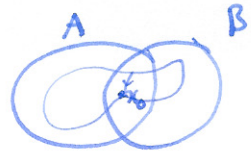
claim: we can apply van Kampen: $A_\alpha \cap A_\beta =$ open nbd of T, contractible $\Rightarrow$ path connected.

$A_\alpha \cap A_\beta \cap A_\gamma$ ~~$=$~~ open nbd of T, also path connected.

so $\pi_1(X) = \ast_\alpha \pi_1(A_\alpha)/N$ $\pi_1(A_\alpha)$ ~~$\ast$~~: $A_\alpha = N_\epsilon(T \cup \alpha) \simeq S^1$. $\pi_1(A_\alpha) \cong \mathbb{Z}$

$\pi_1(A_\alpha \cap A_\beta) \cong \pi_1(N_\epsilon(T)) \cong 1$. so $\pi_1(X) \cong \ast_\alpha \mathbb{Z}$

<u>Proof</u> (of van Kampen) recall:



$X = \bigcup A_\alpha$ $\quad A_\alpha$ open $\quad x_0 \in A_\alpha$ $\quad A_\alpha \cap A_\beta$, $A_\alpha \cap A_\beta \cap A_\gamma$ path connected

$\Phi : * \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$.

<u>claim</u> Φ is surjective.

<u>Proof</u> let $f: I \longrightarrow X$ be a loop <u>claim</u>: there is a partition $0 = s_1 < s_2 < \dots < s_{n+1} = 1$ of $[0,1]$ s.t. for each $[s_i, s_{i+1}]$, $f([s_i, s_{i+1}]) \subset A_\alpha$ for some A_α.

<u>proof</u>: A_α open cover of X, so $f^{-1}(A_\alpha)$ open cover of $[0,1]$, compact, so has a finite subcover. $\square$.

let f_i be $f|_{[s_i, s_{i+1}]}$ and let A_i be ~~the~~ an open set w/ $f([s_i, s_{i+1}]) \subset A_i$.

so $f = f_1 \cdot f_2 \cdots \cdots f_n$ $\quad A_i \cap A_{i+1}$ path connected, so there is a path g_i from x_0 to $f_i(s_{i+1}) = f_{i+1}(s_{i+1})$, so $f \simeq \underbrace{f_1 \cdot \bar{g}_1}_{\subset A_1} \cdot \underbrace{g_1 \cdot f_2 \cdot \bar{g}_2}_{\subset A_2} \cdot \underbrace{g_2 \cdot f_3}_{\subset A_3 \cdots} \cdots \cdot f_n$

so $f \simeq h_1 \cdot h_2 \cdots h_n$ $\quad \in \underset{\alpha \in I}{*} \pi_1(A_\alpha)$

where each $h_i \subset A_i$

so f lies in image of ~~π_1~~ Φ, as required. $\square$.

<u>claim</u> $\ker \Phi = N = \langle i_{\alpha\beta}(f) \, i_{\beta\alpha}(f)^{-1} \rangle$ $\quad$ notation: set $Q = \underset{\alpha}{*} \pi_1(A_\alpha, x_0)/N$.

<u>Notation</u>: A factorization of $[f] \in \pi_1(X, x_0)$ is a formal product $[f_1][f_2] \cdots [f_k]$ where f_i is a loop in (A_α, x_0), $[f_i]$ is the homotopy class of f_i and $f \simeq f_1 \cdot f_2 \cdots f_k$ in X, i.e. $[f_1][f_2] \cdots [f_k]$ is an (unreduced) word in $* \pi_1(A_\alpha, x_0)$ s.t. $\Phi([f_1] \cdots [f_k]) = [f]$. surjectivity of Φ => every $[f]$ has a factorization.

Two factorizations are <u>equivalent</u> if related by the following operations:

1. if adjacent terms $[f_i][f_{i+1}]$ lie in the same A_α, replace them with $[f_i \cdot f_{i+1}]$
2. if f_i is a loop in $A_\alpha \cap A_\beta$ replace $[f_i] \in \pi_1(A_\alpha)$ with $[f_i] \in \pi_1(A_\beta)$.

<u>Remark</u> 1: does not change element of $* \pi_1(A_\alpha, x_0)$

2: does not change image of $[f]$ in $Q = * \pi_1(A_\alpha, x_0)/N$.

so equivalent factorizations give same image in Q.