Useful properties

- $(X,x_0) \xrightarrow{\psi} (Y,y_0) \xrightarrow{\phi} (Z,z_0)$ then $(\phi\psi)_* = \phi_*\psi_*$

- $(X,x_0) \xrightarrow[\substack{id_X \\ 1_X}]{1_X} (X,x_0)$ $\quad (1_X)_* = 1 = id_{\pi_1(X,x_0)}$.

Proof : • associativity $\phi(\psi f) = (\phi\psi)f$.

• $f \mapsto f$ induces an iso $[f] \to [f]$ in $\pi_1(X,x_0)$. $\square$.

Remark : if X,Y homeomorphic $\quad (X,x_0) \underset{\substack{\text{inver} \\ \phi^{-1}=\psi}}{\overset{\phi}{\rightleftarrows}} (Y,y_0)$.

then $\psi\phi = id_X \qquad \phi\psi = id_Y$.

so $(\psi\phi)_* = id_{\pi_1(X,x)} \quad (\phi\psi)_* = id_{\pi_1(Y,y_0)} \Rightarrow \psi_*, \phi_*$ isomorphisms.

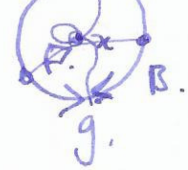
so $\pi_1(X,x_0) \cong \pi_1(Y,y_0)$.

Prop^{n} $\pi_1(S^n) = 0$ if $n \geq 2$.

Proof Let $f: I \to S^n$ be a loop at x_0. If $f(I)$ disjoint from some point $x \in S^n$

then f is a loop in $S^n \setminus \{x\} \cong \mathbb{R}^n$, simply connected, so f null homotopic.

claim : we can homotope f to be non-surjective.



proof (of claim) : Let B be a small open ball around x. ($x_0 \notin B$)

consider segments of $f(I)$ which enter B, hit x and leave B.

$f^{-1}(B)$ open in $[0,1]$ consists of (arbitrary) union of open intervals.

$f^{-1}(x)$ closed $\Rightarrow$ compact in $[0,1]$, connected components of $f^{-1}(B)$ open cover, so has finite subcover

$(a_i,b_i)_{i=1,\dots,N}$. for $f: (a_i,b_i) \to B$ note $f(a_i) \in \partial B$, $f(b_i) \in \partial B$

so choose $g: (a_i,b_i) \to \partial B$ for $g(a_i) = f(a_i)$, $g(b_i) = f(b_i)$

$f|_{(a_i,b_i)} \simeq g|_{(a_i,b_i)}$ by straight line homotopy. as B homeo to subset of $\mathbb{R}^n$.

so $f_0 = f$ homotopic to $f_1 = \begin{cases} f & \text{on } I \setminus \bigcup (a_i,b_i) \\ g_i & \text{on } (a_i,b_i) \end{cases}$ $\quad f_1$ avoids x. $\square$.

Prop^n $\pi_1(S^n) = 0$ if $n \geqslant 2$

Proof Let $f: I \to S^n$ be a loop at x_0. If $f(I)$ disjoint from some point $x \in S^n$, then f is a loop in $S^n \setminus \{x\} \cong \mathbb{R}^n$, simply connected, so f null homotopic.

claim: we can homotope f to be non-surjective

B.

proof (of claim). Let B be a small open ball around x (wlog $x_0 \notin B$)

consider segments of $f(I)$ which enter B, ~~hit x, then~~ and leave B

$f^{-1}(B)$ open in $[0,1]$ consists of a union of open intervals $(a_i, b_i)_{i \in I}$ w/ $f(a_i), f(b_i) \in \partial B$

$f^{-1}(x)$ closed in $[0,1] \Rightarrow$ compact. $f^{-1}(B)$ is an open cover of $f^{-1}(x)$, so has a finite subcover $(a_i, b_i)_{i=1,\dots,N}$. For each (a_i, b_i) choose $g: (a_i, b_i) \to \partial B$ s.t.

$g(a_i) = f(a_i)$ and $g(b_i) = f(b_i)$. $f|_{(a_i,b_i)}$ is homotopic to $g|_{(a_i,b_i)}$ by straight line homotopy, as B homeomorphic to a subset of $\mathbb{R}^n$. So $f \simeq f_0$ is homotopic to $f_1 = \begin{cases} f & \text{on } I \setminus \bigcup (a_i, b_i) \\ g_i & \text{on } (a_i, b_i) \end{cases}$, and f_1 does not hit x. $\square$.

Application $\mathbb{R}^n \setminus \{x\} \simeq S^{n-1}$ so $\pi_1(\mathbb{R}^n \setminus \{x\}) = \mathbb{Z}$ $n = 2$

$= 1$ $n \gneq 2$

Corollary $\mathbb{R}^2$ not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

Induced maps

Prop^n If $r: X \to A$ is a retraction (recall: $i: A \hookrightarrow X$ and $r|_A = id_A$) then $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective. If A is a deformation retraction then i_* is an isomorphism.

Defn $r_t: X \to X$ is a deformation retraction to A if r_t is a homotopy from $r_0 = \mathbb{1}_X$ to $r_1: X \to A$ which is a retraction.

Proof $A \xhookrightarrow{i} X \xrightarrow{r} A$ $\quad r \circ i = id_A$ induces $\pi_1(A, x) \xrightarrow{i_*} \pi_1(X) \xrightarrow{r_*} \pi_1(A)$

so $r_* i_* = \mathbb{1}_{\pi_1(A)} \Rightarrow i_*$ injective, r_* surjective.

now suppose $r_t: X \to X$ is a deformation retraction to A.

let $f: I \to X$ be a loop, then $r_t f: I \to X$ is a homotopy that takes

$r_0 f = \mathbb{1}_X f = f$ to $r_1 f = rf \subset A$ so $[rf] \in \pi_1(A)$ is homotopic to $[f] \in \pi_1(X)$, so $i_* : \pi_1 A \longrightarrow \pi_1 X$ is surjective. $\square$.

Thm [Brouwer fixed point theorem] $S^1 = \partial D^2$ is not a retract of D^2.

Proof suppose there is a retract $S^1 \overset{i}{\hookrightarrow} D^2 \xrightarrow{r} S^1$

$$\pi_1 S^1 \xrightarrow{i_*} \pi_1 D^2 \xrightarrow{r_*} \pi_1 S^1$$

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

i_* not injective
r_* not surjective $\#$ $\square$

Pairs of spaces: (X, A) means $A \subset X$.

$f : (X, A) \to (Y, B)$ cts means $f : X \to Y$ cts and $f(A) \subset f(B)$.

special case: $f : (X, x_0) \to (Y, y_0)$ a map of pairs is called basepoint preserving i.e. $f(x_0) = y_0$.

a homotopy of pairs / basepoint preserving homotopy is a cts family of maps $\phi_t : (X, x_0) \to (Y, x_0)$, i.e. $F : (X \times I, x_0 \times I) \longrightarrow (Y, y_0)$ cts

$$(x, t) \longmapsto \phi_t(x)$$

each f_t induces a map $(\phi_t)_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

$$(f : I \to X) \longmapsto \phi_t \circ f : I \to Y.$$

$$(f : (I, \partial I) \to (X, x_0)) \mapsto (\phi_t f : (I, \partial I) \to (Y, y_0)).$$

Propn $(\phi_0)_* = (\phi_1)_*$ **Proof** $(\phi_0)_*[f] = [\phi_0 f] = [\phi_t f] = [\phi_1 f] = (\phi_1)_*[f]$ $\square$

Defn X, Y are homotopy equivalent if there are maps $X \underset{\psi}{\overset{\phi}{\rightleftarrows}} Y$ s.t. $\phi \circ \psi \simeq id_Y$ and $\psi \circ \phi \simeq id_X$. $(X, x_0), (Y, y_0)$ are homotopy equivalent if there are maps $(X, x_0) \underset{\psi}{\overset{\phi}{\rightleftarrows}} (Y, y_0)$ s.t. $\phi\psi \simeq id_{(Y, y_0)}$ and $\psi\phi \simeq id_{(X, x_0)}$.

i.e. homotopies fix basepoint!

Propn If $\phi : X \to Y$ is a homotopy equivalence, then $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, \phi(x_0))$ is an isomorphism for all $x_0 \in X$.