so $[f_r] \in \pi_1 S^1 \cancel{\cong 1} \cdot [f_r] = 0$ for all r.

pick $r \gg 1 + |a_1| + \dots + |a_n|$

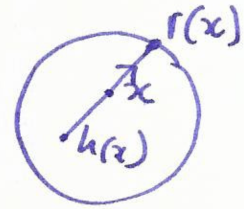
then $|z^n| = r^n = r \cdot r^{n-1} > (|a_1| + \dots + |a_n|)|z^{n-1}| \geq |a_1 z^{n-1} + \dots + a_n|$

consider $p_t(z) = z^n + t(a_1 z^{n-1} + \dots + a_n)$, then this has no roots on the circle $|z| = r$, and for $0 \leq t \leq 1$

$f_r \simeq p_t \simeq p_0$ but $p_0(z) = z^n = [\omega_n] \neq [\omega_0]$ #.

(Brouwer fixed pt thm special case).

<u>Thm</u> Every cts map $h: D^2 \to D^2$ has a fixed point, i.e. there is an $x \in D^2$ s.t. $h(x) = x$.



<u>Proof</u> spose $h(x) \neq x$ for all $x \in D^2$

define $r: D^2 \to S^1$ by setting $r(x)$ to be the intersection of the ray from $h(x)$ thru x with $\partial D^2 = S^1$.

r is a <u>retraction</u> i.e. $r: D^2 \to S^1$ cts and $r(x) = x$ for all $x \in S^1$.

<u>claim</u>: there is no retraction $r: D^2 \to S^1$.

Let f_0 be any loop in S^1. In D^2 f_0 is homotopic, by f_t to the constant loop, e.g. by linear homotopy. But then rf_t is a homotopy from f_0 to constant loop in S^1. #.

<u>Thm</u> If $f: S^2 \to \mathbb{R}^2$ cts, then there is a pair of antipodal points $x, -x$ s.t. $f(x) = f(-x)$ [Borsuk-Ulam in dim 2]

<u>Corollary</u> not injective cts map from $S^2 \to \mathbb{R}^2$.

<u>Proof</u> suppose not, $f: S^2 \to \mathbb{R}^2$ $f(x) \neq f(-x)$ for all x.

define $g: S^2 \to S^1$ by $g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$

let $\alpha: I \to S^2$ be the loop $\alpha(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

then $h = g\alpha: I \to S^1$

<u>note</u>: $g(-x) = -g(x)$ so $h(s+\frac{1}{2}) = -h(s)$ on $s \in [0, \frac{1}{2}]$

lift to $\tilde{h}: [0, \frac{1}{2}] \to \mathbb{R}$ so $\tilde{h}(s+\frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ q odd (~~may~~ depend on s)

q doesn't depend on s: as $q = 2\tilde{h}(s+\frac{1}{2}) - \tilde{h}(s)$ cts on $[0, \frac{1}{2}]$.

so $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$ (odd), so $[h] = q \neq 0 \in \pi_1(S^1) \cong \mathbb{Z}$.

so h <u>not</u> null homotopic. but α null homotopic in S^2 #.

<u>Corollary</u> whenever S^2 is the union of 3 closed sets A_1, A_2, A_3, then at least one contains a pair of antipodal points.

<u>Proof</u> let $d_i: S^2 \to \mathbb{R}$ be distance to A_i $d_i(x) = \inf_{y \in A_i} |x - y|$ cts.

apply Borsuk-Ulam to $\begin{array}{l} S^2 \to \mathbb{R}^2 \\ x \mapsto (d_1(x), d_2(x)) \end{array}$

gives x with $\left.\begin{array}{l} d_1(x) = d_1(-x) \\ d_2(x) = d_2(-x) \end{array}\right\}$ if either zero then $\pm x \in A_1$ or $\pm x \in A_2$; if neither zero then $\pm x \in A_3$. □

π_1 of a product.

<u>Prop^n</u> $\pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y$. if X, Y path connected.

<u>Proof</u> recall product topology $f : Z \to X \times Y$ cts iff $g : Z \to X$ cts
$h : Z \to Y$

where $f(z) = (g(z), h(z))$ $(g = \pi_X f,\ h = \pi_Y f)$

a loop $f : I \to X \times Y$ is therefore a pair of loops $g : I \to X$
$h : I \to Y$

similarly a homotopy $f_t : I \to X \times Y$ gives a pair of homotopies $g_t : I \to X$
$h_t : I \to Y$.

this gives a bijection $\pi_1(X \times Y, (x_0, y_0)) \overset{\Delta}{\longleftrightarrow} \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

this is a group homomorphism: $f_1 \cdot f_2 = (g_1, h_1) \cdot (g_2, h_2)$.
$= (g_1 \cdot g_2, h_1 \cdot h_2)$.

so an isomorphism $\square$.

<u>Example</u> torus $T = S^1 \times S^1$ $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

n-torus $T^n = \underbrace{S^1 \times \dots \times S^1}_{n\text{-times}}$ $\pi_1(T^n) \cong \mathbb{Z}^n$.

<u>Induced homomorphisms</u>

Let $\phi : X \to Y$ cts s.t. $\phi(x_0) = y_0$

<u>claim</u> there is an induced homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$
given by $[f] \longmapsto [\phi \circ f]$

<u>Proof</u> check well defined: suppose $f_0 \simeq f_1$ by f_t then $\phi f_0 \simeq \phi f_1$ by ϕf_t.

· homomorphism $\phi(f \cdot g)$ and $\phi f \cdot \phi g$ are equal as <u>functions</u>.

$$= \begin{cases} \phi f(2s) & 0 \le s \le \frac{1}{2} \\ \phi g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} \quad \square$$