b) for each homotopy $f_t : I \to S^1$ of paths starting at x_0, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lifted homotopy $\tilde{f}_t : I \to \mathbb{R}$ of paths starting at $\tilde{x}_0$.

claim a), b) $\Rightarrow$ Thm.

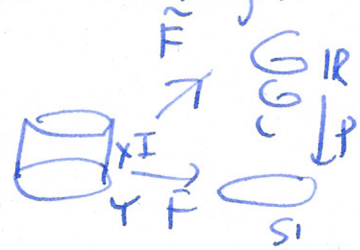
Proof (of claim)

- Φ surjective: let $f : I \to S^1$ be a loop based at $x_0 = (1,0)$ representing an element of $\pi_1(S^1)$. a) $\Rightarrow$ there is a lift $\tilde{f} : I \to \mathbb{R}$ starting at $0 \in \mathbb{R}$. As $p\tilde{f}(1) = x_0$, this implies $\tilde{f}(1) \in p^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. So $\tilde{f}$ is a path in $\mathbb{R}$ from 0 to $n \in \mathbb{Z}$, so is homotopic to ω_n for some n. so $\tilde{f} \simeq \tilde{\omega}_n$ (for some $n(\tilde{f})$) $\Rightarrow$ $p\tilde{f} \simeq \omega_n \Rightarrow [f] = [\omega_n] = \Phi(n)$.

- Φ injective: suppose $\Phi(m) = \Phi(n)$, i.e. $\omega_m \simeq \omega_n$. Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By b) this lifts to a homotopy of paths starting at $0 \in \mathbb{R}$. uniqueness $\Rightarrow$ $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. As $p\tilde{f}_t(1) = x_0 = (1,0)$ for all t, so $\tilde{f}_t(1) \in p^{-1}(x) = \mathbb{Z} \subset \mathbb{R}$ for all t. $\tilde{f}_t$ is map $I \to \mathbb{Z}$ so constant, so $f_0(1) = f_1(1) \Rightarrow m = n$. $\square$.

Proof (of a), b)) we'll prove c):

c) given $F : Y \times I \to S^1$ and a map $\tilde{F} : Y \times \{0\} \to \mathbb{R}$ lifting $F|_{Y \times \{0\}}$ then there is a unique map $\tilde{F} : Y \times I \to \mathbb{R}$ lifting F, and restricting to $\tilde{F}$ on $Y \times \{0\}$.



c) $\Rightarrow$ a): choose $Y = \{pt\}$

c) $\Rightarrow$ b): $f_t(s) \Leftrightarrow F : I \times I \to S^1$
$F(s,t) = f_t(s)$

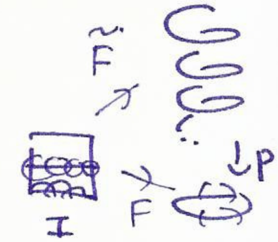
a) implies there is a unique lift $\tilde{F} : I \times \{0\} \to \mathbb{R}$, then c) implies there is a unique lift $\tilde{F} : I \times I \to \mathbb{R}$. note: $\tilde{F}|_{\{0\} \times I}$ and $\tilde{F}|_{\{1\} \times I}$ are lifts of the constant path, and so are constant, so $\tilde{F}$ is a path homotopy.

special property of $p : S^1 \to \mathbb{R}$

(*) there is an open cover $\{U_\alpha\}$ of S^1 such that for each U_α, $p^{-1}(U_\alpha)$ is a disjoint union of sets each of which is homeomorphic to U_α by p.

Example choose $S^1 = [0, 2\pi]/\sim$ $\quad U_1 = (\frac{\pi}{4}, \frac{7\pi}{8})$
$U_2 = (-\frac{3\pi}{4}, \frac{3\pi}{4})$.

Proof (of c) using (*) $\quad F : Y \times I \to S^1$, $\tilde{F} : Y \times \{0\} \to \mathbb{R}$



F cts, so each $(y_0, t) \in Y \times I$ has a product nbd $N_t \times (a_t, b_t)$ s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some single $U_\alpha \subset S^1$.

$\{y_0\} \times I$ compact, so there is a finite cover.

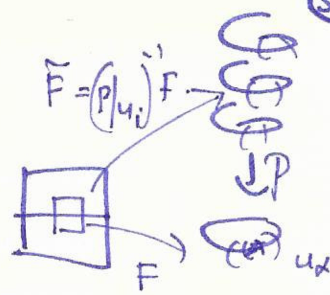
Choose a nbd N of y_0, and a partition $0 \le t_0 < t_1 < \dots < t_n = 1$ of $[0,1]$ such that for each i, $F(N \times (t_i, t_{i+1})) \subset U_i$ for some U_i.

Assume $\tilde{F}$ has been constructed on $N \times [0, t_i]$, then $F(N \times [t_i, t_{i+1}]) \subset U_i$, so by (*) there is an open set $\tilde{U}_i \subset \mathbb{R}$ s.t. $p|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a homeomorphism, and $\tilde{U}_i$ contains $\tilde{F}((y_0, t_i))$.

Now define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ by $\tilde{F} = (p|_{U_i})^{-1} F$

Repeat finitely many times to construct $\tilde{F}$ on $N \times I$.

<u>uniqueness</u>: special case $Y = \{pt\}$



suppose $\tilde{F}, \tilde{F}'$ are two lifts $F : I \to S^1$

such that $\tilde{F}(0) = \tilde{F}'(0)$.

induct on length of partition $0 \le t_0 < t_1 < \cdots < t_n = 1$

so $\tilde{F}(t_i) = \tilde{F}'(t_i)$

$$\begin{tikzcd} & \tilde{U}_\alpha \arrow[d, "p"] \\ {[t_i, t_{i+1}]} \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & U_\alpha \end{tikzcd}$$

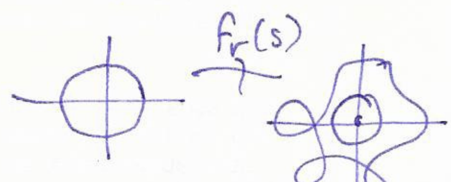
but then

$p|_{U_\alpha}$ homeo $\Rightarrow$ $p\tilde{F} = p\tilde{F}' = F$ $\Rightarrow$ $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$.

<u>Finally</u>: uniqueness on $\{y\} \times I$ $\Rightarrow$ if N, N' overlap then $\tilde{F}|_N = \tilde{F}|_{N'}$ on intersection, so gives lift $\tilde{F} : Y \times I \to \mathbb{R}$. $\square$

<u>Applications</u>

<u>Thm</u> (Fundamental theorem of algebra) Every nonconstant polynomial with coeffs in $\mathbb{C}$ has a root in $\mathbb{C}$.



<u>Proof</u> $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$

If $p(z)$ has no roots in $\mathbb{C}$, then for each $r \in \mathbb{R}$, $r \geq 0$

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$ defines a loop in S^1 based at 1.

f_r is a homotopy of loops based at 1 (check!).

f_0 is the trivial loop $f_0(s) = 1$.