Def^n A loop based at x_0 is a path $f: I \to X$ which starts and ends at x_0

Notation: the set of all homotopy classes of loops based at x_0 is written $\pi_1(X, x_0)$.

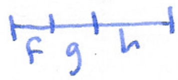
Prop^n $\pi_1(X, x_0)$ is a group wrt the product $[f][g] = [f \cdot g]$
this is called the fundamental group of X with basepoint x_0.

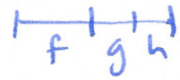
Proof note: $f \cdot g$ make sense as $f(1) = x_0 = g(0)$.
$[f][g] = [f \cdot g]$ is well defined as compositions of homotopic paths are homotopic.
check: group axioms.

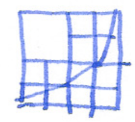
Def^n A reparameterization of a path $f: I \to X$ is a composition $f\phi$ where $\phi: I \to I$ is any cts map w/ $\phi(0) = 0$ and $\phi(1) = 1$.

Note: all reparameterizations are homotopic via $f\phi_t(s) = (1-t)\phi(s) + ts$

Associativity $(f \cdot g) \cdot h$ is a reparameterization of $f \cdot (g \cdot h)$ by

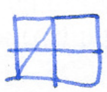




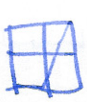


so $(f \cdot g) \cdot h$ is homotopic to $f \cdot (g \cdot h)$, so $([f][g])[h] = [f]([g][h])$

Identity let $c: I \to X$ be the constant path $c(t) = x_0$, then $f \cdot c$, $c \cdot f$, f are reparameterizations of f by



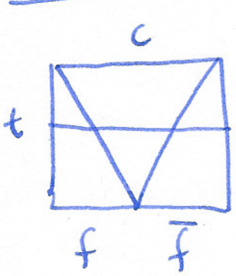
and



so $[c]$ is a two-sided identity in $\pi_1(X, x_0)$.

Inverses let $\bar{f}(t) = f(1-t)$ **claim**: $f \cdot \bar{f}$ homotopic to c.

Proof: use homotopy $h_t = f_t \cdot \bar{g}_t$ where $f_t(s) = \begin{cases} f(s) & 0 \leq s \leq 1-t \\ f(1-t) & s > 1-t. \end{cases}$



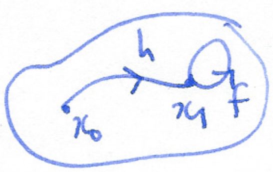
and $g_t(s) = \begin{cases} f(1-t) & s < t \\ f(1-s) & t \leq s \leq 1. \end{cases}$

so $[\bar{f}]$ is a two sided inverse for $[f]$.

Example $X \subseteq \mathbb{R}^n$ convex, then $\pi_1(X, x_0)$ trivial group.

Proof: any two loops $f, g : I \to X$ homotopic by linear homotopy
$f_t(s) = (1-t)f(s) + tg(s)$.

Change of basepoint let $h : I \to X$ be a path from x_0 to x_1



if f is a loop at x_1, then $h \cdot f \cdot \bar{h}$ is a loop at x_0.

Prop$^{\underline{n}}$ $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by
$[f] \longmapsto [h \cdot f \cdot \bar{h}]$ is an isomorphism.

Proof • **well defined**: spose $f_0 \simeq f_1$ by f_t, then $h \cdot f_t \cdot \bar{h}$ is a homotopy from $h \cdot f_0 \cdot \bar{h}$ to $h \cdot f_1 \bar{h}$

• **homomorphism** $\beta_h[f g] = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \bar{h}] = [h f \bar{h}][h \cdot g \bar{h}]$
$= \beta_h[f]\beta_h[g]$.

• **isomorphism** consider $\beta_{\bar{h}} : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$\beta_{\bar{h}}\beta_h([f]) = \beta_{\bar{h}}[h \cdot f \cdot \bar{h}] = [\bar{h} \cdot h \cdot f \cdot h \bar{h}] = [f]$ so $\beta_h = \beta_{\bar{h}}^{-1}$ $\square$.

Def$^{\underline{n}}$ X is **simply connected** if it is path connected and has trivial fundamental group.

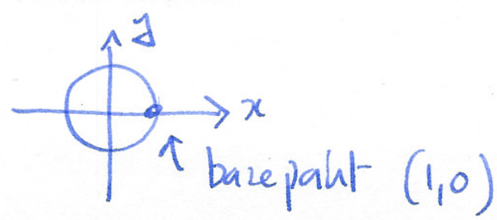
Def$^{\underline{n}}$ X is **path connected** if for any $x, y \in X$ there is a path $f : I \to X$ s.t. $f(0) = x$ and $f(1) = y$

Prop$^{\underline{n}}$ path connectedness is an equivalence relation on points of X. Equivalence classes called **path components**

Warning connected $\not\Rightarrow$ path connected.

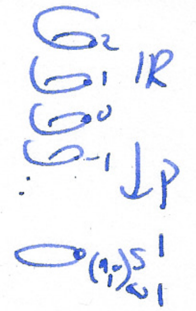
Fundamental group of the circle

S^1 $(x, y) \in \mathbb{R}^2$ s.t. $x^2 + y^2 = 1$
$z \in \mathbb{C}$ $|z| = 1$



examples of loops: $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$

$\theta \longmapsto$ rotation

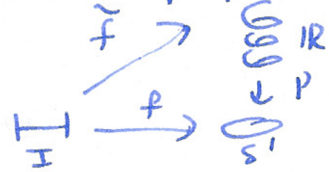


<u>Thm</u> The map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ is an isomorphism.
$n \mapsto [\omega_n(s)]$

<u>Proof</u> Consider the map $p : \mathbb{R} \to S^1$
$t \mapsto e^{2\pi i t}$

<u>note</u> $\omega_n(s) = p\tilde{\omega}_n$ where $\tilde{\omega}_n : I \to \mathbb{R}$
$[0,1] \mapsto [0,n]$
$s \mapsto ns$

<u>Defn</u> if $f : I \to S^1$ is a path, and $\tilde{f} : I \to \mathbb{R}$ is a path s.t. $f = p\tilde{f}$, then we say $\tilde{f}$ is a lift of f.



<u>note</u> we could define $\Phi(n)$ to be $p\tilde{f}$ for $\tilde{f} : I \to \mathbb{R}$ <u>any</u> path from 0 to n, as all such paths are homotopic by the linear homotopy.

<u>check</u> · Φ is a homomorphism (i.e. $\Phi(m+n) = \Phi(m) + \Phi(n)$)

notation: let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation $\tau_m(x) = x + m$

consider $\underbrace{\tilde{\omega}_m}_{\text{path from 0 to } m} \cdot \underbrace{\tau_m \tilde{\omega}_n}_{\text{path from } m \text{ to } m+n} \simeq \underbrace{\tilde{\omega}_{m+n}}_{\text{path from 0 to } m+n}$ (homotopic by linear homotopy)

therefore
$$p(\tilde{\omega}_m \cdot \tau_m \tilde{\omega}_n) \simeq p\tilde{\omega}_{m+n}$$
$$\begin{array}{ccc} \| & & \| \\ p\tilde{\omega}_m \cdot p\tau_m\tilde{\omega}_n & & \Phi(m+n) \\ \| & & \\ p\tilde{\omega}_m \cdot p\tilde{\omega}_n & = & \Phi(m) \cdot \Phi(n) \end{array}$$
, as required $\square$.

<u>· Φ is an isomorphism</u>

<u>useful facts</u>

(i) for each path $f : I \to S^1$ starting at $x_0 \in S^1$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \mathbb{R}$ starting at $\tilde{x}_0$.