Bases finite products: X_i has basis B_i then $X_1 \times \cdots \times X_n$ has bases $\{U_1 \times \cdots \times U_n\}$ $U_i \in B_i$.

infinite products $X = \prod X_i$ $B = \{X_1 \times \cdots \times X_n \times U_{n+1} \times X_{n+2} \times \cdots\}$ $U_{n+1} \in B_{n+1}$ is a subbasis for the topology.

warning $U_1 \times U_2 \times U_3 \times \cdots$ U_i open in X_i, not nec. open in $\prod X_i$!

Thm (Tychonoff) The product of compact spaces is compact. $\square$.

useful fact:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "\pi_i \circ f"'] & \prod X_i \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

f is continuous iff $\pi_i \circ f$ is cts for each X_i.

Connectedness

Defn X is disconnected if $X = U \sqcup V$ disjoint union of two open sets.

Propn If $X = A \cup B$, A, B connected and $A \cap B \neq \emptyset$ then X is connected.

Proof spse $X = U \sqcup V$ open. Suppose $A \cap U = \emptyset$, then $B \cap U \neq \emptyset$ as $A \cap B \neq \emptyset$ $B \cap V \neq \emptyset$, so $B = (B \cap U) \sqcup (B \cap V)$ # B connected. By symmetry $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset \Rightarrow A = (A \cap U) \sqcup (B \cap V) \Rightarrow A$ disconnected. #. $\square$.

Corollary a product of connected sets is connected

Proof $X \times Y$ $\boxplus = \bigcup X \times \{y\} \cup \{x\} \times Y$. $\square$.

Propn The continuous image of a connected set is connected.

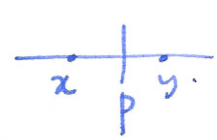
Proof $f: X \to Y$ cts, X connected if $f(X) \subset U \sqcup V$ open, then $f^{-1}(U), f^{-1}(V)$ open in X and $X = f^{-1}(U) \sqcup f^{-1}(V)$ # $\square$.

Propn a subset $A \subseteq \mathbb{R}$ is connected iff A is an interval.

Proof $\Rightarrow$ spse A not an interval, then $\exists\, p \notin A$ s.t. $a < p < b$ with $a, b \in A$ then $A \subset (-\infty, p) \sqcup (p, \infty)$. $\Rightarrow A$ not connected.

$\Leftarrow$ spse A interval and $A = U \sqcup V$, let wlog $\exists\, x \in U, y \in V$ with $x < y$ $p = \sup_{x \in U} \{x \le y\}$

$p \in U$ # sup $p \in V$ # sup. $\square$.



Components A component $C \subset X$ is a maximal connected subset of X

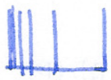
Thm The connected components of X form a partition of X

Proof C_1, C_2 connected and $C_1 \cap C_2 \neq \emptyset \Rightarrow C_1 \cup C_2$ connected. $\square$.

Locally connected

X is locally connected at $x \in X$ if every open set containing x contains a connected open set containing x.

X is locally connected if all points $x \in X$ are locally connected.

Example ~~Hawaiian~~  comb space.

Complete metric spaces

(X,d) metric space. $\{a_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if $\forall \epsilon > 0\ \exists\ N$ s.t. $\forall n, m \geq N$ $d(a_n, a_m) \leq \epsilon$

Prop^n every convergent seq in (X,d) is Cauchy.

Defn A metric space is complete if every Cauchy seq converges.

Example $[0,1]$, $\mathbb{R}$. **Not** $(0,1)$

Completions of metric spaces

Defn $(\overline{X}, d)$ is a completion of (X,d) if $\overline{X}$ is complete, and X is isometric to a dense subset of $\overline{X}$.

Example $\overline{(0,1)} = [0,1]$ $\overline{\mathbb{Q}} = \mathbb{R}$.

Thm A metric space (X,d) has a unique metric completion.

Example $\mathbb{R}^2 \setminus (0,0)$.

Proof (sketch) Let $C[X]$ be the collection of all Cauchy seq in X

define an equivalence relation $(a_n) \sim (b_n)$ if $\lim_{n \to \infty} d_X(a_n, b_n) = 0$

claim $\sim$ is an equivalence relation, define $\overline{X} = C[X]/\sim$

claim define $d_{\overline{X}}((a_n),(b_n)) = \lim_{n \to \infty} d(a_n, b_n)$, this is a metric on $\overline{X}$.

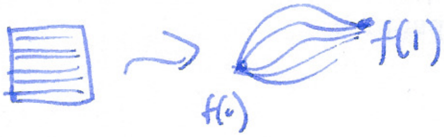
claim $X \hookrightarrow \overline{X}$ $x \mapsto (x)$ constant seq. · injective · isometry · dense image $\square$.

§1 Fundamental group

Motivation (X, x) topological space w/ basepoint. group elements: loops based at x_0, composition  **Example** $(\mathbb{R}^2,(0,0))$  π_1 trivial  $S^1 \vee S^1$ $\pi_1 = F_2$.

§1.1 Paths

Defn A path is a cts $f: I \to X$ $I=[0,1]$ unit interval. A homotopy of paths rel endpoints is a family of paths $f_t: I \to X$ s.t. $f_t(0)=f(0)$ and $f_t(1)=f(1)$, s.t. the corresponding map $F: I\times I \to X$, $(s,t)\mapsto f_t(s)$ is cts.



we say f_0 and f_1 are homotopic (rel endpoints). $f_0 \simeq f_1$.

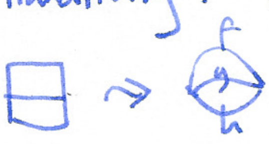
Example Any two paths f_0, f_1 in $\mathbb{R}^n$ w/ same endpoints are homotopic via a linear homotopy $f_t(s) = (1-t) f_0(s) + t f_1(s)$ } cts as multiplication and addition are cts in $\mathbb{R}^n$, and ~~a~~ composition of cts functions is cts.

Propn Homotopy of paths rel endpoints is an equivalence relation $[f]$.

Proof reflexivity: $f \simeq f$ by constant homotopy $f_t(s) = f(s)$

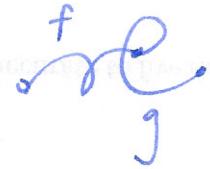
symmetry: $f \simeq g$ by f_t then $g \simeq f$ by f_{1-t}.

transitivity: $f \simeq g$, $g \simeq h$ by say f_t and g_t then define $h_t = \begin{cases} f_{2t} & 0 \le t \le \frac{1}{2} \\ g_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$



key fact. a function defined on the union of two closed sets, which agrees on the overlap, is cts iff cts on each set.

composition of paths: Spse $f: I \to X$, $g: I \to X$ s.t. $f(1) = g(0)$



then there is a path $f\cdot g: I \to X$ defined by $f\cdot g(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

claim composition respects homotopy classes

i.e. if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $f_0 \cdot g_0 \simeq f_1 \cdot g_1$.

Proof given homotopies f_t, g_t define $h_t = f_t \cdot g_t = f_t \cdot g_t(s) = \begin{cases} f_t(2s) & 0 \le s \le \frac{1}{2} \\ g_t(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$ $\square$