Thm The continuous image of a compact set is compact.

Proof $f: X \to Y$. Let U_i be an open cover of $f(X)$, then $f^{-1}(U_i)$ is an open cover of X, so has a finite subcover $f(U_1), \dots f(U_n)$, but then $U_1, \dots U_n$ is a finite cover of $f(X)$ $\square$.

Thm A closed subset of a compact space is compact.

Proof Let $\underset{\text{closed}}{A} \subset \underset{\text{compact}}{X}$. Let U_i be an open cover of A, then $U_i \cup A^c$ is an open cover of X, so has a finite subcover $U_1 \cup A^c, \dots, U_n \cup A^c$, so $U_1, \dots, U_n$ is a finite cover of A. $\square$.

Compactness and Hausdorff spaces

Thm Every compact subset of a Hausdorff space is closed

Proof $\underset{\text{compact}}{K} \subset X$, consider $p \in K^c$ and $q \in K$, $\exists$ disjoint open sets U_p, V_q s.t. $p \in U_p$, $q \in V_q$ and $U_p \cap V_q = \phi$. V_q form an open cover of K, so there is a finite subcover $V_{q_1}, \dots, V_{q_n}$ and $p \in \bigcap U_{q_i} = W$ open, disjoint from $V_{q_1} \cup \dots \cup V_{q_n}$, so $p \in \underset{\text{open}}{W} \subset K^c \Rightarrow K^c$ open $\Rightarrow K$ closed $\square$.

Thm A, B disjoint compact subsets of X, Hausdorff, then there are disjoint open sets U, V, with $A \subset U$, $B \subset V$ $\square$.

Corollary every compact Hausdorff space is normal.

Thm $f: X \to Y$ cts injective, X compact, Y Hausdorff, then X, $f(X)$ are homeomorphic.

Proof suffices to show $f(\text{closed})$ is closed. Let $K \subset X$ be closed, then K is compact, so $f(K)$ compact $\subset f(X)$ compact and Hausdorff $\Rightarrow f(K)$ closed. $\square$.

Sequential compactness

Defn X is sequentially compact if every sequence contains a convergent subsequence.

Local compactness

Def^n X is locally compact if every point X has a compact neighborhood.

Compactifications

Def^n X is embedded in Y, if X is homeomorphic to a subset of Y.

If Y is compact, then Y is a compactification of X.

Examples $\mathbb{R} \cup \{\infty\}$ $\mathbb{R} \cup \{\pm\infty\}$.

One point compactification

(X,T) topological space $\rightsquigarrow$ (X_∞, T_∞) one point compactification

$X_\infty = X \cup \{\infty\}$ $T_\infty = T \cup \{K^c \mid K \text{ compact in } X\}$.

Thm If X is locally compact, Hausdorff, then X_∞ is compact, Hausdorff.

Proof suppose $x, y \in X$ distinct, then $\exists\, U, V$ open $U \cap V = \emptyset$ $x \in U$, $y \in V$.

suppose $x \in X$, $y = \infty \in X_\infty$. Local compactness $\Rightarrow \exists$ compact set K, $x \in K$ neighbourhood, so $\exists\, U$ open $x \in U \subset K$ Now U, K^c open sets separating x, ∞. $\square$.

Compact metric spaces

Thm (X,d) metric space, $A \subset X$. A compact $\Leftrightarrow$ sequentially compact

$\Leftrightarrow$ countably compact

every infinite set has a limit pt

Totally bounded sets

~~Thm~~ Def^n $A \subset (X,d)$ metric space is totally bounded if A has an ϵ-net for every $\epsilon > 0$.

Def^n $A \subset (X,d)$ a finite set of points $x_1, \ldots, x_n$ is an ϵ-net if $A \subset \bigcup B(x_i, \epsilon)$.

Example $H = \ell^2_{\mathbb{R}}(\mathbb{Z})$ $d((a_n),(b_n)) = \sqrt{\sum (a_i - b_i)^2}$

$A = \{e_i\}$ diam$(A) = \sqrt{2}$, no ϵ-net for $\epsilon = \frac{1}{2}$.

Prop^n Totally bounded $\Rightarrow$ bounded

Lemma sequential compactness $\Rightarrow$ totally bounded.

Proof suppose not totally bounded, $\exists\, \epsilon > 0$ s.t. no finite set $\{x_i\}$ has $X \subset \bigcup B(x_i, \epsilon)$, i.e. $\exists$ infinite set s.t. $B(x_i, \epsilon)$ all disjoint, so no convergent sequence. $\square$.

Lebesgue numbers for covers.

(X,d) $A \subset X$ $\{U_i\}$ open cover of A. $\delta > 0$ is a Lebesgue number for $\{U_i\}$ if for all $a \in A$, $B(a, \delta)$ is contained in at least one U_i.

Lemma Every open cover of a sequentially compact subset of a metric space has a Lebesgue number.

Proof $A \subset X$ metric space. seq compact $\{U_i\}$ open cover of A. If no Lebesgue number, then for every $\delta > 0$ there is a $B(a, \delta)$ not contained in a single U_i set $\delta = \frac{1}{n}$, (a_n) has a convergent subsequence $a_n' \to a \in A$, contained in same U_i, but then $\exists\, \epsilon$ s.t. $a \in B(a, \epsilon) \subseteq U_i$ # $\square$.

Thm (X,d) compact $\overset{①}{}\Leftrightarrow$ countably$\overset{②}{}$ compact $\Leftrightarrow$ sequentially$\overset{③}{}$ compact

Proof ① $\Rightarrow$ ② suppose $\{a_i\}$ has no accumulation point, then for every $p \in X$ there is an open set $U_p \cap \{a_i\}$ finite, then U_p from an open cover of X, so have a finite subcover $\Rightarrow \{a_i\}$ finite $\Rightarrow$ constant subsequence has acc pt. $\square$

② $\Rightarrow$ ③ suppose $\{a_n\}$ has accumulation pt p, then every $B(p, \frac{1}{n})$ contains some a_n, so $a_n \to p$. $\square$.

③ $\Rightarrow$ ① sequential compactness $\Rightarrow$ Lebesgue number for open cover $\Rightarrow$ X totally bounded. Let $\{U_i\}$ be an open cover w/ Lebesgue number δ. Then totally bounded $\Rightarrow \exists$ finite set $x_1, \ldots, x_n$, $X \subset B(x_i, \delta)$, but each $B(x_i, \delta) \subset U_i$ for some i, $\Rightarrow$ finite subcover. $\square$.

Product spaces

(X_i, T_i) topological spaces $X = \prod_i X_i$ product set, projection $\pi_i : X \to X_i$

Def^n The product topology is the coarsest topology for which each projection is cts.