<u>Thm</u> X Hausdorff, then every convergent sequence has a unique limit

<u>Thm</u> X 1st countable, then X Hausdorff $\Leftrightarrow$ every convergent sequence has a unique limit

<u>Proof</u> spse $a_n \to a$, $a_n \to b$, let U, V be disjoint open sets containing a, b

as $a_n \to a$ $\exists N$ s.t. $a_n \in U$ for all $n \geq N$, but then $a_n \notin V \Rightarrow a_n \not\to b$ □

<u>Defn</u> X <u>regular</u> if you can separate closed sets and points by disjoint sets, i.e. $\underset{\text{closed}}{F} \subset X$ $p \notin F$, $\exists$ sets U, V open s.t. $F \subset U$, $p \in V$.

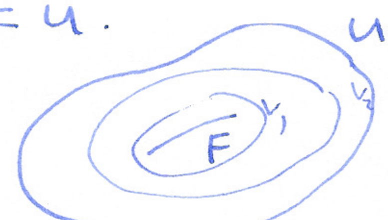
<u>Defn</u> X is T_3 if X is T_1 and regular

<u>Defn</u> X is <u>normal</u> if you can separate any two closed sets by disjoint open sets, i.e. $\underset{\text{closed}}{F_1, F_2} \subset X$, $F_1 \cap F_2 = \phi$, then $\exists U, V$ open w/ $F_1 \subset U$, $F_2 \subset V$.

<u>Defn</u> X is T_4 iff X is T_1 and normal

<u>Example</u> metric spaces are T_4. <u>Remark</u> $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

<u>Thm</u> X is normal iff : for every closed set F and every open set U $F \subset U$, there is an open set V with $F \subset V \subset \overline{V} \subset U$.



<u>Proof</u> $\Rightarrow$ $\underset{\text{closed}}{F} \subset \underset{\text{open}}{U}$ U^c closed, and $F \cap U^c = \phi$

normal $\Rightarrow$ $\exists$ disjoint open sets V_1, V_2 with $F \subset V_1$, $U^c \subset V_2$

<u>claim</u> : $F \underset{\checkmark}{\subset} V_1 \underset{\checkmark}{\subset} \overline{V}_1 \subset U$ <u>note</u>: $V_1 \subset V_2^c$ closed so $\overline{V}_1 \subset \underset{\text{closed.}}{V_2^c} \subset U$. □

$\Leftarrow$ F_1, F_2 disjoint closed sets, then $F_1 \subset F_2^c$ open, so there is open U w/ $F_1 \subset U \subset \overline{U} \subset F_2^c$, so U and $\overline{U}^c$ disjoint open sets containing F_1, F_2 □.

Thm (Urysohn's Lemma) X normal, F_1, F_2 disjoint closed subsets, then there is a cts function $f: X \to [0,1]$ s.t. $f(F_1) = 0$ and $f(F_2) = 1$.

Proof $F_1 \cap F_2 = \emptyset$ so $F_1 \subset F_2^C$, so by above there is $U_{1/2}$ open with

$F_1 \subset U_{1/2} \subset \overline{U_{1/2}} \subset F_2^c$, apply again: $F_1 \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_{3/4} \subset \overline{U_{3/4}} \subset F_2^c$

etc. let D = dyadic fractions $\frac{p}{2^n}$ (lowest terms) in $[0,1]$, gives open sets U_d, $d \in D$, s.t. if $d_1 < d_2$, then $\overline{U_{d_1}} \subset U_{d_2}$.

define $f(x) = \begin{cases} \inf \{ d \mid x \in U_d \} & x \notin F_2 \\ 1 & x \in F_2 \end{cases}$

claim f is cts: want $f^{-1}(\text{open})$ is open.

suffices to check for a subbasis set: $[0,a)$, $(b,0]$ is a subbase for $[0,1]$

note: $f^{-1}([0,a)) = \{ \bigcup U_d \mid d < a \}$ open. $\square$.

Thm (Urysohn) Every 2nd countable $\underbrace{\text{normal } T_1}_{= T_4}$ space is metrizable.

Proof (sketch) X 2nd countable T_4, homeomorphic to a subset of the Hilbert cube $I \subset \mathbb{R}^\infty$: $(a_1, a_2, \dots)$ s.t. $0 \leq a_n \leq \frac{1}{n}$

Hilbert space: $(a_1, a_2, \dots)$ $\sum a_n^2 < \infty$, metric $d(p,q) = \sqrt{\sum_{n=1}^{\infty} |a_n - b_n|^2}$

let B be a countable basis $\{B_1, B_2, \dots\}$, assume $X \in B$.

normal $\Rightarrow$ for each B_i there is B_j s.t. $\overline{B_j} \subset B_i$

consider all such (countably many) pairs. p_n: $B_{j_n} \subset B_{i_n}$.

Urysohn's Lemma: if there is a function $f_n: X \to [0,1]$ s.t.

$f_n(\overline{B_{j_n}}) = 0$ and $f_n(B_{i_n}^c) = 1$

now define $f: X \to I$ by $f(x) = \left(\frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \frac{f_3(x)}{2^3}, \dots \right)$

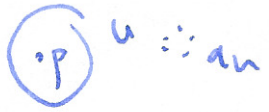
claim: f injective: the f_n separate points

so $\exists f_n$ s.t. $f_n(x) = 0$, $f_n(y) = 1$.

<u>claim</u> f cts at $p \in X$: given $\epsilon > 0$, there is an open set U with $\|f(p) - f(q)\|^2 \leq \epsilon^2$ for all $q \in U$ ← only need to worry about first $n \sim \log_2(1/\epsilon)$ functions, each cts. so $\exists$ open subsets $U_1, \dots, U_n$ with this property.

<u>claim</u> f^{-1} cts: show if $a_n \not\to p$ then $f(a_n) \not\to f(p)$.

$a_n \not\to p$, so there is an open set $p \in U$ containing finitely many a_n, pass to subsequence, contains no terms of a_n. so there is a fixed pair



P_m : $p \in \bar{B}_{jm} \subset B_{jn}$ with $f_m(p) = 0$, $f_m(a_n) = 1$

so $\|f(a_n) - f(p)\|^2 \geq \frac{1}{2^m}$ $\forall n$. $\square$

<u>Functions that separate points</u>

F collection of functions $\{f : X \to \mathbb{R}\}$ <u>separates pts</u> if $\forall a, b \in X$ $\exists f \in F$ with $f(a) \neq f(b)$.

<u>Prop^n</u> If $C(X, \mathbb{R})$ separates pts $\Rightarrow$ X is Hausdorff
↑ cts function from X to $\mathbb{R}$

<u>Def^n</u> X <u>completely regular</u> if for all F closed $\subset X$, $p \notin F$, there is $f : X \to [0,1]$ cts, s.t. $f(p) = 0$, $f(F) = 1$.

<u>Prop^n</u> completely regular $\Rightarrow$ regular

<u>Th^m</u> $\mathcal{C}(X, \mathbb{R})$ separates points for a completely regular T_1 space.

<u>Compactness</u>

An open cover of X is a collection of open sets s.t. $X = \bigcup U_i$

<u>Def^n</u> $A \subset X$ is <u>compact</u> if every open cover has a finite subcover

<u>Th^m</u> (Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded

<u>Examples</u> not $(0,1)$, $\mathbb{R}$.