Last time

Defn. (X, T) topological space. $B \subset T$ is a basis if every open set in T is a union of elements of B.

Subspaces / relative topology

(X, T) topological space, $A \subset X$, then relative topology on A is $T_A = \{ U \cap A \mid U \in T \}$.

Exercise: check T_A is a topology on A.

Example: $[0,1] \subseteq \mathbb{R}$, $[0, \frac{1}{2})$ is open in relative topology.

Quotient topology (X, T) top. space $\sim$ equivalence relation on X

$q : X \to X/\sim$, $x \mapsto [x]$ quotient topology on $X/\sim$ is $A \subset X/\sim$ is open iff $q^{-1}(A)$ open in X.

Example · $I = [0,1]$ $0 \sim 1$ 

· $\mathbb{R}$, $x \sim y$ if $|x - y| \in \mathbb{Q}$ $\mathbb{R}/\sim$ has open sets ϕ, $\mathbb{R}/\sim$.

Countability: a set A is countable is there is a bijection to a subset of $\mathbb{Z}$.

bad example: long line

X is first countable if for all $p \in X$ there is a countable nbd base B_p at p

not: $(\mathbb{R}$, cofinite top$)$ $(\mathbb{R}$, cocountable top.$)$

(X, T) is second countable if T has a countable basis.

not: $(\mathbb{R}$, discrete$)$, long line.

Prop^n 2nd countable $\Rightarrow$ 1st countable $\square$.

Thm (Lindelhof 1) $A \subset X$ 2nd countable, any open cover of A has a countable subcover

Thm (Lindelhof 2) X 2nd countable, every basis contains a countable basis.

Separable spaces

Def^n X is separable if X contains a countable dense subset

not: $(\mathbb{R}, \text{discrete})$ $\ell^\infty(\mathbb{N})$

Prop^n 2nd countable $\Rightarrow$ separable

Proof B_n countable basis, choose $b_n \in B_n$, claim $\{b_n\}$ dense in X $\square$.

Thm X metric space, separable $\Rightarrow$ 2nd countable.

Proof let A be a countable dense set in X, let B be $\{B(a,q) \mid a \in A, q \in \mathbb{Q}\}$

claim B is a basis. **Proof** $x \in U$ open, $\exists\, r$ s.t. $x \in B(x,r) \subset U$

choose $a \in B(x,r)$ with $d(a,x) < \frac{r}{4}$, then $x \in B(a,q) \subset B(x,r)$ for $\frac{r}{4} < q < \frac{r}{2}$ $\square$.

Remark $A \subset X$ 2nd countable $\Rightarrow$ A 2nd countable

$A \subset X$ separable $\not\Rightarrow$ A separable.

Separation

Def^n (X,T) is T_1 if for every pair of distinct points $a,b \in X$, there are open sets U,V with $a \in U$, $b \notin U$; $b \in V$, $a \notin V$

Warning: $U \cap V \neq \emptyset$ always, is possible

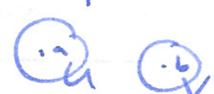
not: $X = \{0,1\}$ $T = \{\emptyset, \{0\}, X\}$

Thm X is T_1 iff each point set $\{x\}$ is closed.

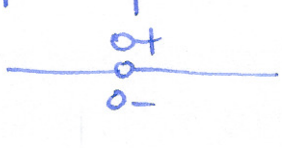
Proof $\Rightarrow$ show $\{p\}^c$ open. If $q \in \{p\}^c$ then $\exists$ open set U with $q \in U$ $p \notin U$ $\Rightarrow$ $q \in U \subset \{p\}^c$ $\Rightarrow$ $\{p\}^c$ open.

$\Leftarrow$ $\{a\}, \{b\}$ closed, then $\{a\}^c, \{b\}^c$ open $\square$.

Def^n (X,T) is T_2 or Hausdorff if for all distinct pair of points a,b there is a disjoint pair of open sets w/ $a \in U$, $b \in V$.



Non-example



open sets $(a,b) \not\ni 0$

$(a,b) \setminus 0 \cup 0+$

$(a,b) \setminus 0 \cup 0-$

$/\sim$

$(x,0) \sim (x,1)$ $x \neq 0$