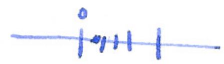
<u>Observation</u> if $A \subset X$ and every $x \in A$ has an open set U s.t. $x \in U \subset A$ then A is open.

<u>Warning</u> (X,d) $\mathcal{T} = \{ B(x,r) \mid x \in X, r > 0 \}$ not nec. a topology.

<u>Limits / accumulation points</u>

X topological space $A \subset X$, a point $p \in X$ is an <u>accumulation point</u> of A if every open set containing p intersects $A \setminus p$. 

<u>Example</u> • $A = (0,1) \subset \mathbb{R}$ has accumulation points $[0,1]$ 

• $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ has accumulation points $\{0\}$. 

• $\mathbb{Z} \subset \mathbb{R}$ <u>no</u> accumulation points.

<u>Thm</u> [Bolzano-Weierstrauss] If $A \subset \mathbb{R}$ is a bounded infinite set, then A has at least one accumulation point.

<u>Closed sets</u>

X topological space, $A \subset X$ is <u>closed</u> iff $X \setminus A = A^c$ = complement of A is open.

<u>Examples</u> $[0,1] \subset \mathbb{R}$, $\{0\} \subset \mathbb{R}$, any finite collection of points in $\mathbb{R}$.

<u>Non-examples</u> $(0,1)$, $(0,1]$ ← not open or closed

<u>Examples</u> $\emptyset, X$ both open and closed (sometimes called clopen).

(X, discrete topology) every set both open and closed.

<u>Thm</u> X topological space, then the closed sets satisfy:

1) $\emptyset, X$ closed
2) arbitrary intersection of closed sets is closed
3) finite unions of closed sets are closed □

<u>Thm</u> $A \subset X$ is closed iff A contains its set of accumulation points.

<u>Proof</u> $\Rightarrow$ A closed then A^c open. $p \in A^c$ open $\Rightarrow \exists$ open set U s.t. $p \in U \subset A^c$ so $p \notin Acc(A)$, so $Acc(A) \subseteq A$.

$\Leftarrow$ suppose $Acc(A) \subset A$. <u>claim</u> A^c is open. <u>Proof</u> let $p \in A^c$, then $p \notin Acc(A)$, so $\exists$ open set U s.t. $p \in U \subseteq A^c$ with $A \cap U = \phi$, so A^c open $\Rightarrow A$ closed $\square$

Closure of a set

$A \subset X$, $\bar{A}$ closure of A is the intersection of all closed sets containing A.

Prop^n 1) $\bar{A}$ is closed

2) A is closed iff $A = \bar{A}$.

Proof 1) arbitrary intersection of closed sets is closed.

2) $\Leftarrow$ from 1). $\Rightarrow$ $A \in \{$all closed sets containing $A\}$ so $\bigcap_{\substack{\text{closed sets}\\ \text{containing } A}} = A$. $\square$

Thm $\bar{A} = A \cup acc(A)$

Proof spose $x \notin A\cup Acc(A)$, then $\exists$ open set U s.t. $x \in U$ and $U \cap A = \phi$, so $x \notin U^c$, closed, contains A, so $x \notin \bar{A}$, so $\bar{A} \subset A \cup Acc(A)$.

$x \notin \bar{A} \Rightarrow$ open set $x \in U \subseteq A^c$, $U \cap A = \phi \Rightarrow x \notin Acc(A)$ $\square$.

Examples $\overline{(0,1)} = [0,1]$ in $\mathbb{R}$ $\quad \bar{\mathbb{Q}} = \mathbb{R}$ in $\mathbb{R}$.

warning $\overline{B(x,r)} \neq \{y \in X \mid d(x,y) \leq r\}$!.

Def^n $A \subset X$ is <u>dense</u> in B if $B \subset \bar{A}$

$x \in A$ lies in the <u>interior</u> of A if $\exists$ open set U s.t. $x \in U \subseteq A$ notation A°

Prop^n A° is open, A° is the largest open subset of A, A is open iff $A^\circ = A$ $\square$

Def^n the <u>exterior</u> of A, $ext(A)$ is $int(A^c)$

the <u>boundary / frontier</u> of A is $bound(A) = b(A) = X \setminus (A^\circ \cup ext(A))$.

Thm $\bar{A} = A^\circ \cup b(A)$

Proof recall $\bar{A} = A \cup acc(A)$. ~~If $x \in \mathring{A}$, then $x \in ext(A) \Rightarrow$~~ $\overset{\text{open}}{U} \subseteq A$

$\supseteq$ $\mathring{A} \subseteq \bar{A}$, spose $x \in b(A) = X \setminus (A^\circ \cup ext(A)) \Rightarrow$ every open set $x \in U$ contains pts of both $A, A^c \Rightarrow x \in acc(A)$.

$\subseteq$ $x \in \bar{A} \setminus A^\circ \Rightarrow$ every open set containing x contain points of A, and $A^c \Rightarrow x \in b(A)$ $\square$.

Examples $(0,1) \subseteq \mathbb{R}$ $\quad int(0,1) = (0,1)$ $\quad \overline{(0,1)} = (0,1)$ $\quad b(0,1) = \{0\} \cup \{1\}$.

$\mathbb{Q} \subseteq \mathbb{R}$. $\bar{\mathbb{Q}} = \mathbb{R}$ $\quad int(\mathbb{Q}) = \phi$. $\quad b(\mathbb{Q}) = \mathbb{R}$. $\quad$ warning $b(B(x,r)) \neq \{y \mid d(x,y) = r\}$!

Def^{n} $A \subseteq X$ is <u>nowhere dense</u> if $\text{int}(\bar{A}) = \emptyset$.

Examples $\mathbb{Z} \subseteq \mathbb{R}$ $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \subseteq \mathbb{R}$, Cantor set.

Def^{n} A <u>neighbourhood</u> N for $p \in X$ is a set N s.t. there is an open set $p \in U \subseteq N$.

The set of all neighbourhoods N_p is called the <u>neighbourhood system</u> of p.

Remark neighbourhoods need not be open or closed.

<u>Neighbourhood properties / axioms</u>

1) $N_p \neq \emptyset$, for each $N \in N_p$, $p \in N$.
2) If $A, B \in N_p$ then $A \cap B \in N_p$
3) If $A \in N_p$ and $A \subset B$, then $B \in N_p$
4) each $A \in N_p$ contains a nbd $B \in N_p$ s.t. B is a neighbourhood for each point $x \in B$.

<u>Convergent sequences</u>

A sequence $(a_n)_{n \in \mathbb{N}}$ converges to a point $b \in X$, written $\lim_{n \to \infty} a_n = b$ or $a_n \to b$ if for every open set U, with $b \in U$, $\exists N$ s.t. $a_n \in U$ for all $n \geq N$.

Remark in a metric space can define a <u>Cauchy sequence</u>, i.e. $\{a_n\}$ s.t. $\forall \epsilon > 0$ $\exists N$ s.t. $|a_n - a_m| \leq \epsilon$ for all $m, n \geq N$, and we say a metric space is <u>complete</u> if every Cauchy seq. converges.

<u>Comparing topologies</u>

Let T_1, T_2 be topologies on X. Spse $T_1 \subset T_2$ (i.e. every open set in T_1 is open in T_2) then we say T_1 is <u>smaller</u>, coarser, weaker than T_2

T_2 is <u>larger</u>, finer, stronger than T_1.

Note collection of topologies is partially ordered by inclusion

Example trivial $\subseteq$ cofinite $\subseteq$ discrete.