Topology I Math 70700 Room 5417 ①

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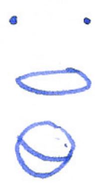
<u>Motivation</u>

youtube downfall version

Q: when are two spaces the same? e.g. 

find (algebraic) invariants - if they're different the spaces are different.

e.g. # connected components



S^0: 2 components

S^1: 1 component

S^2: 1 - component

higher dimensional connectedness:

simply connected: every map $S^1 \to X$ can be shrunk to a point, e.g. disc S^2.

not simply connected: e.g S^1.

two versions: homotopy groups (non-abelian)

homology groups (abelian) ← computable

<u>Plan</u>: 1. Review of general topology

2. Fundamental group

3. Homology

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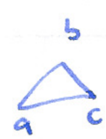
1. General topology

<u>Motivation</u>: $f: \mathbb{R} \to \mathbb{R}$ continuous (at x)

<u>Def^n</u> If for all $\epsilon > 0$ there is a $\delta > 0$ s.t. for all y $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$

This definition works in any metric space

<u>Def^n</u> (X,d) is a metric space if there is $d: X \times X \to \mathbb{R}$ s.t.

- $d(a,b) \geq 0$ and $d(a,a) = 0$
- $d(a,b) = d(b,a)$ (symmetry)
- $d(a,b) + d(b,c) \geq d(a,c)$ (triangle inequality) 
- $d(a,b) = 0 \Rightarrow a = b$ (properness?).

Examples · $(\mathbb{R}, |\cdot|)$ $\quad d(a,b) = |a-b|$

· $(\mathbb{R}^2, |\cdot|)$ $\quad d((a_1,a_2),(b_1,b_2)) = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2}$

$$= \max\{|a_1-b_1|, |a_2-b_2|\}$$

· trivial metric $\quad d(a,b) = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{otherwise} \end{cases}$

· restriction metric on subset $Y \subset X$, $\quad d_Y(a,b) = d_X(a,b)$

warning not induced path metric!

· $C([0,1])$ cts functions on $[0,1]$ $\quad d(f,g) = \int_0^1 |f(x)-g(x)|\,dx$

or $\quad d(f,g) = \sup|f(x)-g(x)|$

Continuous function, better definition

$f: X \to Y$ cts if for every open set $A \subset Y$, $f^{-1}(A)$ is open in X.

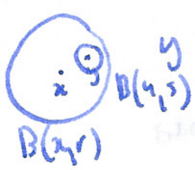
open sets (metric space version)

$B(x,r)$ = open ball of radius r about x, i.e. $\{y \in X \mid d(x,y) < r\}$

in $\mathbb{R}$, these are just open intervals. (a,b)

Defn A set $U \subset X$ is open if for every $x \in U$ there is an r s.t. $B(x,r) \subset U$.

Example · $B(x,r)$ is open

check: 

$y \in B(x,r)$, choose $B(y,s)$ with $s = \frac{r-d(x,y)}{2}$

show $B(y,s) \subset B(x,r)$: spose $z \in B(y,s)$ then

$$d(x,z) \leq d(x,y) + d(y,z) = d(x,y) + \frac{r-d(x,y)}{2} = \frac{r+d(x,y)}{2} < r \quad \square$$

· empty set $\emptyset \subset X$ is open, $X \subseteq X$ is open.

- in $\mathbb{R}$: (a,b) open as are $(0,\infty)$, $\mathbb{R}$

not open: $[0,1]$, $(0,1]$, $[0,\infty)$, $\{0\}$.

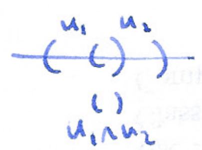
<u>Key properties</u>

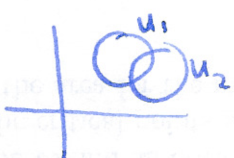
<u>Thm</u> 1) the union of any collection of open sets is open

2) the intersection of any finite number of open sets is open

<u>Example</u> $\bigcap (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ not open in $\mathbb{R}$

<u>Proof</u> 1) Pf $x \in \{U_i\}_{i \in I}$ then $x \in U_i$ for some i, so $\exists\, B(x,r) \subset U_i \subset \bigcup U_i$ $\square$.

2) <u>Examples</u> 



$B(x,r) \cap B(y,s)$ not a ball in general.

<u>Pf</u> $x \in U_1 \cap U_2$, then $\exists\, r_1, r_2$ s.t. $B(x,r_1) \subset U_1$, $B(x,r_2) \subset U_2$ so $B(x, \min(r_1,r_2)) \subset U_1 \cap U_2$ $\square$

<u>Exercise</u> $f:(X,d_X) \to (Y,d_Y)$, show $f^{-1}(\text{open})$ open equivalent to ϵ,δ version of cts.

<u>General case</u> (X not metric space)

<u>Defn</u> X set $(\neq \emptyset)$. A <u>topology</u> on X is a collection of subsets T s.t.

1) $\emptyset, X \in T$

2) an arbitrary union of elements of T lies in T

3) finite intersections of elements of T lie in T

<u>Examples</u> ($\mathbb{R}$, standard topology)

($\mathbb{R}$, trivial topology) $T = \{\emptyset, \mathbb{R}\}$. (aka indiscrete topology)

($\mathbb{R}$, discrete topology) T = all subsets.

(X, cofinite topology) T = sets w/ finite complement, $\cup\, \emptyset$.

$(X,T_1), (X,T_2)$ topologies, then $(X, T_1 \cup T_2)$ is a topology.

<u>Remark</u> $T_1 \cup T_2$ not nec. a topology <u>Example</u> $X = \{1,2,3\}$

[Zariski topology, Gromov-Hausdorff topology] $T_1 = \emptyset, \{1\}, X$ $T_2 = \emptyset, \{2\}, X$.