Math 301 Introduction to Proofs Fall 19 Midterm 2

Name: ____________________________

- You may use a 3×5 index card of notes.

(1) (a) Prove that for all integers x, if x^2 is even, then x is even.
(b) Prove that $\sqrt[2]{2}$ is irrational. (You may use part (a).)

(2) Give examples of:
(a) a function $f\colon \mathbb{R} \to \mathbb{R}$ which is injective but not surjective.
(b) a function $f\colon \mathbb{Z} \to \mathbb{N}$ which is surjective but not injective.

(3) What do the following statements mean in ordinary language?
(a) $(\forall x \in \mathbb{R})(\exists a \in A)(x < a)$
(b) $(\exists x \in \mathbb{R})(\forall a \in A)(x < a)$
(c) $(\forall A \subseteq \mathbb{R})(\exists x \in \mathbb{R})(\forall a \in A)(x < a)$

(4) State the negation of the following statements, using appropriate quantifiers:
(a) The function $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing.
(b) The integer n is only divisible by powers of 3.

(5) Write out a careful proof or give a counterexample to the following statement:
For any pair of consecutive integers, the sums of their squares is odd.

(6) Write out a careful proof or give a counterexample to the following statement:
Let $f\colon X \to Y$ be a function, and let $A, B \subseteq X$. Then $f(A \cap B) = f(A) \cap f(B)$.

(7) Write out a careful proof or give a counterexample to the following statement:
If $g \circ f$ is injective then f is injective.

(8) Write out a careful proof or give a counterexample to the following statement:
If $g \circ f$ is injective then g is injective.

(9) Consider the statement:
If $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing then it is injective.
(a) State the contrapositive of the statement, and then prove or give a counterexample.
(b) State the converse of the statement, and then prove or give a counterexample.

(10) Let $f\colon X \to Y$ be a function. Prove that for any $A, B \subseteq Y$, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

MT2 Solutions

Q1 a) **Thm** If x^2 is even, then x is even $(x \in \mathbb{Z})$.

Proof (contrapositive) Suppose x is odd. Then $x = 2n+1$ for some $n \in \mathbb{Z}$. Then $x^2 = (2n+1)^2 = 4n^2+4n+1 = 2(2n^2+2n)+1$, odd, which implies x^2 is odd. $\square$.

b) **Thm** $\sqrt{2}$ is irrational. **Proof** (by contradiction) Suppose $\sqrt{2} = a/b$, $a,b \in \mathbb{Z}$, no common factor. Then $2 = a^2/b^2$, so $2b^2 = a^2$, so $2 \mid a^2$. a) $\Rightarrow$ $2 \mid a$ so $a = 2n$ for some $n \in \mathbb{Z}$. Then $2b^2 = (2n)^2 = 4n^2 \Rightarrow b^2 = 2n^2 \Rightarrow 2 \mid b^2$. a) $\Rightarrow$ $2 \mid b$. But then a,b have common factor 2. # $\square$.

Q2 a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.

b) $f : \mathbb{Z} \to \mathbb{N}$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \nearrow & \nearrow & & & \\ 0,1 & -1 & 2 & -2 & \cdots \end{array}$$

or $f(x) = |x| + 1$

Q3 a) A has no upper bound.

b) A has a lower bound.

c) Every subset of $\mathbb{R}$ has a lower bound.

Q4 a) $f : \mathbb{R} \to \mathbb{R}$ strictly increasing.

$(\forall x,y \in \mathbb{R})(x > y \Rightarrow f(x) > f(y))$.

negation: $(\exists x,y \in \mathbb{R})(x > y \text{ and } f(x) \leq f(y))$.

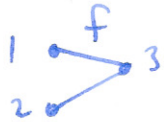
b) $(\forall a \in \mathbb{Z})(a \mid n \Rightarrow \exists m \in \mathbb{Z} \text{ s.t. } a = 3^m)$.

negation $(\exists a \in \mathbb{Z})(a \mid n \text{ and } \forall m \in \mathbb{Z},\ a \neq 3^m)$.

Q5 **Thm** For any pair of consecutive integers, the sums of their squares are odd.

Proof Let $n \in \mathbb{Z}$, and consider $n^2 + (n+1)^2 = n^2 + n^2 + 2n + 1$
$= 2n^2 + 2n + 1 = 2(n^2+n) + 1$, which is odd. $\square$.

Q6 False



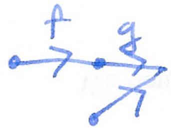
$A = \{1\}$
$B = \{2\}$
$A \cap B = \emptyset$.

$f(\emptyset) = \emptyset$

$f(A) = f(B) = \{3\}$ $\therefore$ $f(A) \cap f(B) = \{3\} \neq \emptyset$.

Q7 **Thm** If $g \circ f$ is injective, then f is injective. $A \xrightarrow{f} B \xrightarrow{g} C$

Proof (contrapositive) Suppose f is not injective, then $\exists\, x, y \in A$ s.t. $x \neq y$ and $f(x) = f(y)$. but then $g(f(x)) = g(f(y))$ and $x \neq y \Rightarrow g \circ f$ not injective. $\square$.

Q8 False.  $g \cdot f$ injective but g not injective.

Q9 a) If f is not injective, then f is not strictly increasing

Proof If f is not injective, then $\exists\, x, y \in \mathbb{R}$ s.t. $f(x) = f(y)$ and $x \neq y$. wlog $x < y$, but $f(x) = f(y)$ $\not\Rightarrow$ f strictly increasing $\square$.

b) If f is injective then f is strictly increasing. False $f(x) = -x$.

Q10 $f : X \to Y$, $A, B \subseteq Y$.

Thm $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Proof $\subseteq$ Let $x \in f^{-1}(A \cup B)$ then $f(x) \in A \cup B$, so $f(x) \in A$ or $f(x) \in B$. This implies $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ so $x \in f^{-1}(A) \cup f^{-1}(B)$.

$\supseteq$ Let $x \in f^{-1}(A) \cup f^{-1}(B)$. Then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$, so $f(x) \in A$ or $f(x) \in B$, so $f(x) \in A \cup B \Rightarrow x \in f^{-1}(A \cup B)$. $\square$.