SMT2 Solutions

Q1 Thm $\sqrt[3]{2}$ is irrational.

Proof (by contradiction) suppose $\sqrt[3]{2} = \frac{a}{b}$, for $a,b \in \mathbb{N}$ and fraction in lowest terms. Then $2 = a^3/b^3$ so $2b^3 = a^3$ so $2 | a^3$. Claim if $2|a^3$ then $2|a$. Proof (of claim) (by contradiction) suppose $2 \nmid a$ then $a = 2n+1$ for some $n \in \mathbb{N}$. Then $a^3 = (2n+1)^3 = 8n^3 + 12n^2 + 6n + 1$, which is odd $\#$ $\square$. Therefore $a = 2n$ for some $n \in \mathbb{N}$, so $2b^3 = (2n)^3 = 8n^3 \Rightarrow b^3 = 4n^3, \Rightarrow 2|b$ $\#$ a,b in lowest terms. $\square$.

Q2 $f: \mathbb{N} \to \mathbb{Z}$ a) $n \mapsto n+1$. b) $n \mapsto (n-2)^2$.

Q3 $f: \mathbb{R} \to \mathbb{R}$ b) $x \mapsto (x-1)x(x+1)$ b) $x \mapsto x^2$.

Q4 a) $x \mapsto \frac{1}{x}$ b) $x \mapsto x-1$ c) $x \mapsto -x$ d) composition of these: $x \mapsto -(\frac{1}{x} - 1)$

Q5 Thm If A, B countable, then $A \times B$ countable.

Proof Claim: there is an injective map $\mathbb{N} \times \mathbb{N} \xrightarrow{\phi} \mathbb{N}$ Proof

	1	2	3
1	(1,1)	(1,2)	(1,3) ...
2	(2,1)	(3,2)	(4,3) ...
3	(3,1)	(3,2)	(3,3) ...
	$\vdots$	$\vdots$	$\vdots$

$\square$.

Claim if $f: A \to \mathbb{N}$ injective and $g: B \to \mathbb{N}$ injective then $f \times g: A \times B \to \mathbb{N} \times \mathbb{N}$ injective. Proof suppose $f\times g(a,b) = f\times g(c,d)$ then $(f(a), g(b)) = (f(c), g(d)) \Rightarrow f(a) = f(c)$ and $g(b) = g(d)$, f, g inj $\Rightarrow a = c$ and $b = d$. $\square$

therefore composition $A \times B \to \mathbb{N}$ $\phi \cdot (f \times g)$ is injective. $\square$.

Q6 a) x is not a power of 3. negation: $(\exists n \in \mathbb{N})(x = 3^n)$.

b) $\sqrt{2}$ is rational. negation: $(\forall a,b \in \mathbb{N})(\sqrt{2} \neq a/b)$

c) $f: A \to B$ surjective. negation: $(\exists b \in B)(\forall a \in A)(f(a) \neq b)$.

d) $\forall n \in \mathbb{N}$, $n^3 - n$ is even. negation: $(\exists n \in \mathbb{N})(2 \nmid n^3 - n)$.

e) there is $A \subseteq \mathbb{R}$ not bounded above negation: $(\forall A \subset \mathbb{R})(\exists x \in \mathbb{R})(\forall a \in A)(x \leq a)$.

f) The set of primes is not bounded above. negation: $(\exists n \in \mathbb{N})(\forall p \in \mathbb{N})(p \leq n)$ or $(\exists q \in \mathbb{N})(q | p)$ and $(q \neq 1)$ and $(q \neq p)$).

Q7 a) $(\exists a,b \in \mathbb{N})(\pi^2 = a/b)$.

b) $f: A \to B$, $(\exists a,b \in A)(f(a) = f(b)$ and $a \neq b)$ and $(\forall b \in B)(\exists a \in A)(f(a) = b)$.

c) Let $P = \{p \in \mathbb{N} \mid$ if $q \in \mathbb{N}$ and $q | p$ then $q = 1$ or $q = p\}$ then either $n \in P$ or $(\exists p, q, r \in P)($ $p \neq q$, $p \neq r$ and $q \neq r$ and $p|n$, $q|n$ and $r|n$).

d) $(\forall f:A\to B)(\exists a,b\in A)(f(a)=f(b) \text{ and } a\neq b)$.

e) $f:\mathbb{R}\to\mathbb{R}$ $(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(f(y)>x)$ and $(\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(f(y)>x)$.

f) $\lim_{x\to 1} f(x)=0$ means $(\forall\epsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R})(|x-1|<\delta \Rightarrow |f(x)|<\epsilon)$.

negation: $(\exists\epsilon>0)(\forall\delta>0)(\exists x\in\mathbb{R})(|x-1| \overset{\leq}{\cancel{<}} \delta \text{ and } |f(x)|\geq\epsilon)$.

Q8 a) True sum of squares of any 3 consecutive integers, +1 is divisible by 3.

Proof let $n\in\mathbb{Z}$, consider $n^2+(n+1)^2+(n+2)^2+1 = n^2+n^2+2n+1+n^2+4n+4+1$
$=3n^2+6n+6 = 3(n^2+2n+2)$, so divisible by 3. □.

b) False: $1^2+2^2+3^2+4^2=1+4+9+16=30$, not divisible by 4.

c) True $f:A\to B$ has an inverse iff surjective and injective.

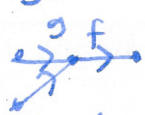
Proof: try and define $f^{-1}(b)=a$, where $f(a)=b$.

$\Rightarrow$ suppose $f:A\to B$ has an inverse $f^{-1}:B\to A$. Then $\forall b\in B\ \exists a$ s.t. $f^{-1}(b)=a$
$\Rightarrow b=f(a)$, so f is surjective. Suppose $f(x)=f(y)$, then $f^{-1}(f(x))=f^{-1}(f(y))$
so $x=y \Rightarrow f$ injective.

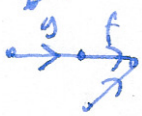
$\Leftarrow$ If f surjective, then $\forall b\in B, \exists a\in A$ s.t. $f(a)=b$, so $f^{-1}(\{b\})$ has at least one element. Claim: $f^{-1}(\{b\})$ contains one element, which we can call $f^{-1}(b)$.

Proof (by contradiction) suppose $a,c\in f^{-1}(\{b\})$ and $a\neq c$, then $f(a)=f(b)=f(c)$ but $a\neq c$
$\#$ f injective. D. so f^{-1} well defined. □.

d) false.



e) false



f) false: $g(x)=-x$ decreasing $f(x)=x$ increasing $f\circ g(x)=-x$ decreasing

g) ~~false~~ true: Thm if $x^2\leq x$ ~~then~~ $x\leq 1$. Proof (contrapositive) Suppose $x>1$ then
$x=1+a$, $a>0$. So $x^2=(1+a)^2=1+2a+a^2=\underbrace{(1+a)}_{=x}+\underbrace{(a+a^2)}_{>0} = x+\underbrace{(a+a^2)}_{>0}$ so $x^2>x$. □.

Q9 $f:X\to Y$, $A\subseteq Y$ then $f^{-1}(A)=\{x\in X \mid f(x)\in A\}\subseteq X$, so $f^{-1}:P(Y)\to P(X)$.
as $f^{-1}(A)$ defined for all subsets $A\subseteq Y$, and $f^{-1}(A)$ depends only on A.

injective: f^{-1} injective iff ~~f injective and~~ f surjective ~~i.e. bijection~~.

Proof suppose f^{-1} injective, then $f^{-1}(A)=f^{-1}(B)\Rightarrow A=B$.

spse f not surjective, then $\exists\, y \in Y$ s.t. $f^{-1}(\{y\}) = \emptyset$, but $f^{-1}(\emptyset) = \emptyset$. $\#$.

spse f surjective, but $\exists\, A \neq B$ s.t. $f^{-1}(A) = f^{-1}(B)$, wlog $A, B \neq \emptyset$, so $\exists$ wlog $\exists\, a \in \cancel{A \setminus B}$. and $x \in X$ s.t. $f(x) = a$, but then $x \in f^{-1}(A)$ but $x \notin f^{-1}(B)$ $\#$. $\square$.

surjective f^{-1} surjective iff f injective.

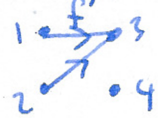
Proof $\Rightarrow$ spse f not injective, then $\exists\, a, b \in X$ s.t. $f(a) = f(b)$, $a \neq b$. Then $f^{-1}(f(a)) = \{a, b\}$ Claim $\nexists\, A \subseteq Y$ s.t. $f^{-1}(A) = \{a\}$. spse $f^{-1}(A) = \{a\}$, then $f(a) \in A \Rightarrow b \in f^{-1}(A)$ as $f(b) = f(a)$ $\#$. $\square$.

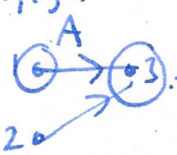
$\Leftarrow$ suppose f injective, but f^{-1} not surjective. note $f^{-1}(\emptyset) = \emptyset$, so $\exists\, A \subseteq X$ s.t. $\nexists\, B \subseteq Y$ s.t. $f^{-1}(B) = A$ Claim: $f^{-1}(f(A)) = A$. Proof: if $a \in A$ then $f(a) \in f(A)$, so $a \in f^{-1}(f(A))$. suppose $a \notin A$ but $a \in f^{-1}(f(A))$ then $f(a) \in f(A)$ so $\exists\, b \in A$ s.t. $f(a) = f(b)$ f injective $\Rightarrow a = b$ $\#$ $\square$ so f^{-1} surjective. $\square$.

Q10 $f: X \to Y$ $A \subseteq X$, $B \subseteq Y$

a) False:  $A = \{1\}$ $f(A) = \{3\}$ $f^{-1}(f(A)) = f^{-1}(\{3\}) = \{1, 2\} \neq \{1\}$.

b) True $B \subseteq f(f^{-1}(B))$. Proof let $b \in B$, ~~then~~ and let $x \in X$ s.t. $f(x) = b$, so $x \in f^{-1}(B)$, then $f(x) \in \cancel{B}$ $f(f^{-1}(B))$, ~~so $f(f^{-1}(B))$~~ $f(x) = b \in f(f^{-1}(B))$ $\square$.

c) True $f(A \cup f^{-1}(B)) = f(A) \cup B$. ~~Proof $\supseteq$ suppose $y \in f(A) \cup B$, then $y \in f(A)$ or $y \in B$ if $y \in f(A)$ then $\exists\, a \in A$ s.t. $f(a) = y$, so $a \in A \cup f^{-1}(B)$, so $y \in f(A \cup f^{-1}(B))$. If $y \in B$, then $\exists\, x \in X$ s.~~ False:  $A = \{1\}$ $B = \{4\}$. $f^{-1}(B) = \emptyset$ $f(A) \cup B = \{3, 4\}$ so $f(A \cup f^{-1}(B)) = f(A \cup \emptyset) = f(A) = \{3\} \neq \{3, 4\}$.

d) False:  $f(A) = B$. $A = \{1\}$ $B = \{3\}$. $A \cap f^{-1}(B) = \{1\}$. $f^{-1}(f(A) \cap B) = f^{-1}(\{3\}) = \{1, 2\} \neq \{1\}$.