Math 301 Introduction to Proof Fall 19 Final

Name: Solutions

- **Start each question on a fresh sheet of paper. Staple together in numerical order at the end of the exam.**
- **You may use a 3x5 index card of notes.**

(1) Write out a detailed proof of the fact that the sum of any integer with its square is even.

(2) Prove that $\sqrt{2}$ is irrational.

(3) Consider the statement:
If $x > y$ then $x^2 > y^2$.
Which, if any, of the following substitutions give a counterexample?
(a) $x = 2, y = -1$ (b) $x = 1, y = -2$ (c) $x = -2, y = -1$

(4) Indicate which of the following statements are true, vacuously true, or false.
(a) $(A \cap B)' = A' \cap B'$.
(b) If p is a natural number which is both prime and a square, then p is equal to 1.
(c) If $\mathcal{P}(A) = \{\varnothing\}$ then $A = \varnothing$.

(5) Write out the following statements using quantifiers. Then write out their negations.
(a) The function $f\colon A \to B$ is injective.
(b) There is an injective function $f\colon A \to B$.

(6) Prove or give a counterexample: If $g \circ f$ is surjective and f is surjective, then g is surjective.

(7) Let $f\colon X \to Y$ be a function, and let $A \subseteq X$. Prove or give a counterexample to the following statements:
(a) $f^{-1}(f(A)) \subseteq A$.
(b) $A \subseteq f^{-1}(f(A))$.

(8) State the contrapositive of "If $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing, then it is injective." Prove or give a counterexample.

(9) Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

(10) Show that $n^2 \leqslant 2^n$ for all $n \geqslant 4$.

(11) Show that if x is a real number and $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer.

(12) Define an equivalence relation on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \sim (c, d)$ if $ad = bc$. Show that $\sim$ is an equivalence relation. What are the equivalence classes?

Q1 Thm For any integer n, n^2+n is even.

Proof Suppose n is even, then $n=2a$ for some $a \in \mathbb{Z}$. Then $n^2+n = (2a)^2+2a = 2(2a^2+a)$ which is even. Otherwise n is odd, so $n=2a+1$ for some $a \in \mathbb{Z}$. Then $n^2+n = (2a+1)^2+(2a+1) = 4a^2+4a+1+2a+1 = 4a^2+6a+2 = 2(2a^2+3a+1)$, even $\square$

Q2 Thm $\sqrt{2}$ is irrational.

Proof (by contradiction) Suppose $\sqrt{2} = a/b$, $a,b \in \mathbb{Z}$, fraction in lowest terms. Then $2 = a^2/b^2$ so $2b^2 = a^2$, so $2 \mid a^2$. Claim if $2 \mid a^2$ then $2 \mid a$. Proof (of claim) (contrapositive) suppose a is odd, then $a = 2n+1$ for some $n \in \mathbb{Z}$, so $a^2 = (2n+1)^2 = 4n^2+4n+1$, odd $\square$. So $a = 2c$ for some $c \in \mathbb{Z}$, so $2b^2 = (2c)^2 = 4c^2$, so $b^2 = 2c^2$, so $2 \mid b^2 \Rightarrow 2 \mid b$, # a,b no common factor. $\square$.

Q3 b) is counterexample.

Q4 a) F b) vacuously true c) T.

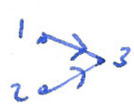
Q5 a) $(\forall x,y \in A)(f(x)=f(y) \Rightarrow x=y)$ negation: $(\exists x,y \in A)(f(x)=f(y) \text{ and } x \neq y)$

b) $(\exists f: A \to B)(\forall x,y \in A)(f(x)=f(y) \Rightarrow x=y)$

negation: $(\forall f: A \to B)(\exists x,y \in A)(f(x)=f(y) \text{ and } x \neq y)$.

Q6 Thm If $g \circ f$ is surjective and f is surjective, then g is surjective.

Proof Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. Let $z \in Z$. As $g \circ f$ is surjective, there is an $x \in X$ such that $g(f(x)) = z$, but then $f(x) \in Y$ and $g(f(x)) = z$, so for every $z \in Z$, there is $y = f(x) \in Y$ s.t. $f(y) = z$, so g is surjective. $\square$.

Q7 $f: X \to Y$, $A \subseteq X$. a) false :  $A = \{1\}$, $f(A) = \{3\}$, $f^{-1}(f(A)) = \{1,2\} \not\subseteq A$

b) Thm $A \subseteq f^{-1}(f(A))$.

Proof Let $a \in A$, then $f(a) \in f(A)$, as $f(a) = f(a) \in f(A)$, $a \in f^{-1}(f(A))$ so $A \subseteq f^{-1}(f(A))$ $\square$

Q8 Thm If $f: \mathbb{R} \to \mathbb{R}$ is not injective, then it is not strictly increasing.

Proof If $f: \mathbb{R} \to \mathbb{R}$ is not injective, then there are $x,y \in \mathbb{R}$ s.t. $f(x) = f(y)$ and $x \neq y$. wlog $x < y$, and $f(x) = f(y)$ # $f(x) < f(y)$ needed for strictly increasing $\square$.

Q9 Thm $1+3+5+\cdots+(2n-1)=n^2$

Proof (induction) base case $n=1$: $1=1^2$ ✓.

induction step: $p(n): 1+3+\cdots+(2n-1)=n^2$.

Consider $\underbrace{1+3+5+\cdots+(2n-1)}_{=n^2 \text{ by } p(n)}+(2n+1) = n^2+2n+1=(n+1)^2$, so $p(n+1)$ holds. $\square$.

Q10 Thm $n^2 \leq 2^n$ for $n \geq 4$.

Proof (induction) base case $n=4$: $4^2=16 \leq 2^4=16$ ✓.

induction step: $p(n): n^2 \leq 2^n$

consider $\frac{(n+1)^2}{n^2} \leq 2 \iff (n+1)^2 \leq 2n^2 \iff n^2+2n+1 \leq 2n^2 \iff 2n+1 \leq n^2$.

as $2n+1 \leq 3n$, this holds for $n \geq 3$.

Therefore $n^2 \leq 2^n$ and $\frac{(n+1)^2}{n^2} \leq 2$ $(n \geq 3)$ so $n^2 \cdot \frac{(n+1)^2}{n^2} \leq 2 \cdot 2^n$, so $\underbrace{(n+1)^2 \leq 2^{n+1}}_{p(n+1)}$ $\square$.

Q11 Thm $x \in \mathbb{R}$, $x+\frac{1}{x} \in \mathbb{Z}$, then $x^n+\frac{1}{x^n} \in \mathbb{Z}$.

Proof (induction) base case $n=1$ $x+\frac{1}{x} \in \mathbb{Z}$ by hypothesis.

induction step: suppose $p(n): x^n+\frac{1}{x^n} \in \mathbb{Z}$. Consider $\underbrace{\left(x^n+\frac{1}{x^n}\right)}_{\substack{\in \mathbb{Z} \\ (p(n))}}\underbrace{\left(x+\frac{1}{x}\right)}_{\substack{\in \mathbb{Z} \\ p(1)}} = x^{n+1}+x^{n-1}+\frac{1}{x^{n-1}}+\frac{1}{x^{n+1}}$

$= x^{n+1}+\frac{1}{x^{n+1}} + \underbrace{x^{n-1}+\frac{1}{x^{n-1}}}_{\text{hold by } p(n-1)}$ Note: for $p(1) \overset{?}{\Rightarrow} p(2)$, $p(n-1)=p(1)$ so this works here. $\boxtimes$.

$\Rightarrow x^{n+1}+\frac{1}{x^{n+1}} \in \mathbb{Z}$, giving $p(n+1)$ $\square$.

Q12 $\mathbb{N}\times\mathbb{N}$ ~~should be $\mathbb{N}\setminus\{0\}$~~ with $(a,b)\sim(c,d)$ if $ad=bc$.

reflexive: $(a,b)\sim(a,b)$ as $ab=ab$ ✓.

symmetric: $(a,b)\sim(c,d) \Rightarrow ad=bc$, so $bc=ad$, so $(c,d)\sim(a,b)$ ✓.

transitive: $(a,b)\sim(c,d)$, $(c,d)\sim(e,f) \Rightarrow ad=bc$ and $cf=de \Rightarrow adcf=bcde$

$\Rightarrow af(dc)=be(cd) \Rightarrow af=be$ as long as $cd \neq 0$, true as $\in \mathbb{N}$ so $(a,b)\sim(e,f)$

equivalence classes: $ad=bc \iff \frac{a}{b}=\frac{c}{d}$, equivalent fractions. $\square$.