Math 301 Introduction to Proof Fall 19 Sample Final

(1) Consider the statement
If x and y are real numbers with $x < y$ then $\frac{1}{x} > \frac{1}{y}$.
Which, if any, of the following substitions gives a counterexample:
(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = -2, y = -1$

(2) State which of the following statements are true, vacuously true, or false.
(a) $(A \cup B)' = A' \cup B'$
(b) If $\mathcal{P}(A) = \varnothing$, then $A = \varnothing$
(c) $(A \cap B) - C = (A - C) \cap (B - C)$

(3) Let $f \colon X \to Y$ be a function, and let A and B be subsets of Y. Show that $f(A \cup B) = f(A) \cup f(B)$.

(4) Let $f \colon X \to Y$ be a function, and let A be a subset of Y. Show that $f(f^{-1}(A)) \subset A$. Is $f(f^{-1}(A)) = A$?

(5) Either give examples of functions with the properties below, or explain why they don't exist.
(a) A surjective function from $\mathbb{Z}$ to $[0, 1]$
(b) An injective function from $(0, \infty)$ to $(0, 1)$
(c) A surjective function from $[0, 1]$ to $[0, 1] \times [0, 1]$

(6) State the negation of the following statements, using appropriate quantifiers.
(a) The function $f \colon A \to B$ is surjective.
(b) There are no injective functions $f \colon A \to B$.
(c) The function $f \colon \mathbb{R} \to \mathbb{R}$ is increasing.

(7) Write out careful proofs, or give counterexamples, to the following statements.
(a) If n is an integer then $n^3 - n^2$ is even.
(b) If g and $g \circ f$ are surjective, then g is surjective.
(c) If $f \colon \mathbb{R} \to \mathbb{R}$ is decreasing, and $g \colon \mathbb{R} \to \mathbb{R}$ is decreasing, then $f + g$ is decreasing.

(8) Write out the negation of the statement, "$(\exists x)(\sim p(x))$ or $(\forall x)(q(x))$".

(9) Write out the converse to the statement "If $p(x)$ and $q(x)$ then $r(x)$".

(10) Write out the contrapositive to the statement "If $p(x)$ and $q(x)$ then $r(x)$".

(11) We say a sequence a_n has a limit L if for all $\epsilon > 0$ there is an N such that $|L - a_n| \leq \epsilon$ for all $n \geq N$. Write this statement out using the quantifier symbols, then write out the negation of this statement. Use the negation to show that the sequence $a_n = (-1)^n$ does not have a limit.

(12) Show that the following numbers are irrational: $\sqrt{3}, \sqrt{5}, \sqrt{15}, \sqrt{3} - \sqrt{5}$.

(13) Show that $1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = (n-1)2^{n+1} + 2$.

(14) Show that $n^3 \leq 3^n$ for $n \geq 3$.

(15) Show that $5^n + 2 \times 11^n$ is divisible by 3.

(16) Show that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all integers x, y and all natural numbers n.

(17) Show that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1$.

(18) Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

(19) Consider the relation on $\mathbb{R}$ determined by $x \sim y$ if $x - y$ is rational. Is this an equivalence relation?

(20) Consider the relation on functions $f \colon \mathbb{R} \to \mathbb{R}$ given by $f \sim g$ if there are numbers $A, B \in \mathbb{R}$ such that $f(x) \leq Ag(x) + B$. Is this an equivalence relation?

(21) Define a relation on sets by $A \sim B$ if there is an injection $f \colon A \to B$. Is this an equivalence relation on sets?

(22) Let F be the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$. Define a relation on F by $f \sim g$ if $g = f'$. Is this an equivalence relation? Does it define a function?