Sample final Solutions

Q1 Counterexample: c)

Q2 a) false b) vacuously true c) true.

Q3 **Thm** $f: X \to Y$, $A, B \subseteq X$ $\quad f(A \cup B) = f(A) \cup f(B)$

Proof $\subseteq$: suppose $y \in f(A \cup B)$, then there is $x \in A \cup B$ s.t. $f(x) = y$. So $x \in A$ or $x \in B$, so $f(x) \in f(A)$ or $f(x) \in f(B)$, so $f(x) \in f(A) \cup f(B)$. As $y = f(x)$, $y \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$ $\square$

$\supseteq$: let $y \in f(A) \cup f(B)$, then there is $x \in A$ st. $f(x) = y$, or there is $x \in B$ st. $f(x) = y$. Therefore, there is $x \in A \cup B$ s.t. $f(x) = y$ which implies $f(x) \in f(A \cup B)$. As $y = f(x)$, this gives $y \in f(A \cup B)$, so $f(A) \cup f(B) \subseteq f(A \cup B)$ $\square$.

Q4 **Thm** $f: X \to Y$, $A \subseteq Y$. then $f(f^{-1}(A)) \subseteq A$.

Proof suppose $y \in f(f^{-1}(A))$. then there is $x \in f^{-1}(A)$ s.t. $f(x) = y$. As $x \in f^{-1}(A)$, $f(x) \in A$. As $y = f(x)$ this implies $y \in A$, as required.

$f(f^{-1}(A)) \neq A$. Example: $1 \longmapsto \cdot 2$, $\cdot 3$ $\quad A = \{2, 3\}$ $\quad f^{-1}(A) = \{1\}$ $\quad f(f^{-1}(A)) = \{2\}$.

Q5 a) no such function $[0,1]$ is uncountable, $\mathbb{Z}$ is countable.

b) $(0,\infty) \longmapsto (1,\infty) \longmapsto (0,1)$ so $f(x) = \frac{1}{x+1}$ works.
$\quad x \longmapsto x+1 \quad x \longmapsto \frac{1}{x}$

c) send $0.a_1a_2a_3\ldots \longmapsto (0.a_1a_3a_5\ldots, 0.a_2a_4a_6\ldots)$

Q6 a) not($f: A \to B$ surjective) is $(\exists b \in B)(\forall a \in A)(f(a) \neq b)$.

b) not(there are no injective functions $f: A \to B$) is $(\exists f: A \to B)(\forall a, b \in A)(f(a) = f(b) \Rightarrow a = b)$.

c) not($f: \mathbb{R} \to \mathbb{R}$ is increasing) is $(\exists x, y \in \mathbb{R})(x < y)(f(x) \geq f(y))$

Q7 a) **Thm** $n \in \mathbb{Z}$, then $n^3 - n^2$ is even

Proof If n is even then $n = 2a$ for some $a \in \mathbb{Z}$, and $n^3 - n^2 = 8a^3 - 4a^2$, even.
If n is odd, then $n = 2a+1$ for some $a \in \mathbb{Z}$ and $n^3 - n^2 = (2a+1)^3 - (2a+1)^2 = 8a^3 + 12a^2 + 6a + 1 - 4a^2 - 4a - 1 = 8a^3 + 8a^2 + 2a$ even. $\square$.

b) If g and $g \circ f$ are surjective, then f is surjective. false: 

c) Thm If $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are decreasing, then $f+g$ is decreasing ②

Proof f,g decreasing means $\forall x,y\in\mathbb{R}$, $x<y \Rightarrow f(x)\geq f(y)$
and $\forall x,y\in\mathbb{R}$ $x<y \Rightarrow g(x)\geq g(y)$. $\Big\}$ (*).

but then $(f+g)(x) = f(x)+g(x)$ and $(f+g)(y) = f(y)+g(y)$

so (*) $\Rightarrow$ $f(x)+g(x)\geq f(y)+g(y)$ so $f+g$ is decreasing $\square$.

Q8 $(\forall x)(p(x))$ and $(\exists x)(\sim q(x))$.

Q9 $r(x) \Rightarrow p(x)$ and $q(x)$.

Q10 not $r(x) \Rightarrow$ (not $p(x)$) or (not $q(x)$).

Q11 $a_n \to L$ means $(\forall \epsilon>0)(\exists N\in\mathbb{N})(\forall n\geq N)(|L-a_n|\leq\epsilon)$.

negation: $(\exists\epsilon>0)(\forall N\in\mathbb{N})(\exists n\geq N)(|L-a_n|>\epsilon)$.

Thm $a_n=(-1)^n$ does not have a limit.

Proof choose $\epsilon=\frac{1}{2}$, then for any N, pick $n\geq N$ n even. then $(-1)^n=1$, $(-1)^{n+1}=-1$.
as $|1-(-1)|=2$, for any $L\in\mathbb{R}$ either $|L-1|>\frac{1}{2}$ or $|L+1|>\frac{1}{2}$ as required. $\square$.

Q12 Thm $\sqrt{3}$ is irrational.

Proof (by contradiction) suppose $\sqrt{3}=\frac{a}{b}$ in lowest terms, then $3=a^2/b^2$, so $3b^2=a^2$.
Claim: $3|n^2 \Rightarrow 3|n$ Pf (of claim) (by contrapositive) suppose $3\nmid n$, then $n=3a+1$ or $n=3a+2$, so $n^2=(3a+1)^2=9a^2+6a+1$ or $n^2=(3a+2)^2=9a^2+12a+4$, in either case $3\nmid n^2$ $\square$.
claim $\Rightarrow 3|a$, so $a=3c$ for some $c\in\mathbb{N}$. Then $3b^2=(3c)^2=9c^2 \Rightarrow b^2=3c^2 \Rightarrow$
$3|b^2$ claim $\Rightarrow 3|b$, # a,b no common factor. $\square$

Thm $\sqrt{5}$ is irrational.

Pf (by contradiction) suppose $\sqrt{5}=a/b$ in lowest terms, then $5=a^2/b^2$, so $5b^2=a^2$.
Claim: $5|n^2 \Rightarrow 5|n$. Pf (of claim) (contrapositive) suppose $5\nmid n$, then $n=p_1^{n_1}\cdots p_k^{n_k}$, p_i primes and no $p_i=5$. then $n^2=p_1^{2n_1}\cdots p_k^{2n_k}$ and no $p_i=5$, unique prime factorization $\Rightarrow 5\nmid n^2$. $\square$. claim $\Rightarrow 5|a$, so $a=5c$ for some $c\in\mathbb{N}$. Then $5b^2=(5c)^2=25c^2$
$\Rightarrow b^2=5c^2 \Rightarrow 5|b^2$, claim $\Rightarrow 5|b$ # a,b in lowest terms. $\square$.

Thm $\sqrt{15}$ is irrational.

Pf (by contradiction) Suppose $\sqrt{15} = a/b$ in lowest terms, then $15 = a^2/b^2$, so $15b^2 = a^2$.
claims above $\Rightarrow$ ($3 | a^2$, $\Rightarrow$ $3 | a$.) so $a = 3c$ for some c. then $15b^2 = (3c)^2 = 9c$
so $5b^2 = 3c$ but then $3 | 5b^2$, unique factorization $\Rightarrow 3 | b^2$, claim $\Rightarrow 3 | b$, # lowest terms. $\square$

Thm $\sqrt{3} - \sqrt{5}$ is irrational.

Proof (by contradiction) suppose $\sqrt{3} - \sqrt{5} = a/b$ in lowest terms. Then $(\sqrt{3} - \sqrt{5})^2 = a^2/b^2$.
$3 - 2\sqrt{15} + 5 = a^2/b^2$ so $\sqrt{15} = -1 - \frac{a^2}{2b^2}$, rational, # $\sqrt{15}$ irrational. $\square$

Q13 Thm $1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = (n-1) 2^{n+1} + 2$

Proof (induction) ① base case $n=1$: $1 \times 2 = (1-1) 2^0 + 2 = 2$ ✓

② induction step: assume $p(n)$: $1 \times 2 + \cdots + n \times 2^n = (n-1) 2^{n+1} + 2$, and consider

$\underbrace{1 \times 2 + 2 \times 2^2 + \cdots + n \times 2^n}_{= (n-1) 2^{n+1} + 2 \text{ by } p(n)} + (n+1) 2^{n+1} = (n-1) 2^{n+1} + 2 + (n+1) 2^{n+1}$

$= (n - 1 + n + 1) 2^{n+1} + 2 = (2n) 2^{n+1} + 2 = n 2^{n+2} + 2$, so $p(n+1)$ holds. $\square$

Q14 Thm $n^3 \leq 3^n$ for $n \geq 3$.

Proof (induction) base case: $n=3$: $3^3 = 3^3$ ✓.

induction step: assume $p(n)$: $n^3 \leq 3^n$

claim: $\frac{(n+1)^3}{n^3} \leq 3$ for $n \geq 3$. Pf (of claim): for $n > 0$ $\frac{(n+1)^3}{n^3} \leq 3 \Leftrightarrow (n+1)^3 \leq 3n^2$.

$\Leftrightarrow n^3 + 3n^2 + 3n + 1 \leq 3n^2 \Leftrightarrow 3n^2 + 3n + 1 \leq 2n^3$. note $n \leq n^2$ and $1 \leq n^2$, so.
$3n^2 + 3n + 1 \leq 7n^2$ and $7n^2 \leq 2n^3$ for $n \geq 0$ iff $\frac{7}{2} \leq n$. so this holds for $n \geq 4$.
Do $n = 3$ by hand: $3n^2 + 3n + 1 = 37$, $2n^3 = 54$ ✓. $\square$ claim.

$p(n)$: $n^3 \leq 3^n$ and $\frac{(n+1)^3}{n^3} \leq 3 \Rightarrow \frac{n^3 (n+1)^3}{n^3} \leq 3.3^{n+1} \Rightarrow (n+1)^3 \leq 3^{n+1}$,
so $p(n+1)$ holds. $\square$

Q15 Thm $3 | 5^n + 2 \times 11^n$.

Proof base case $n=1$: $5 + 22 = 27$, $3 | 27$ ✓.

induction step: assume true for $p(n)$, so $3 | 5^n + 2 \times 11^n$, and consider

$5^{n+1} + 2 \times 11^n = 5.5^n + 2.11.11^n = (6-1) 5^n + 2.(12-1) 11^n = \underbrace{6.5^n + 2.12.11^n}_{= 3(2.5^n + 2.4.11^n)} - \underbrace{(5^n + 2.11^n)}_{3 | \text{ by } p(n)}$

so $3 | 5^{n+1} + 2 \times 11^n$ by $p(n)$, so $p(n+1)$ holds $\square$.

Q16 **Thm** $x^{2n}-y^{2n}$ is divisible by $x+y$ for all $x,y\in\mathbb{Z}$ $n\in\mathbb{N}$.

Proof (induction on n) base cases: $n=1$ $x^2-y^2=(x+y)(x-y)$ ✓.

$n=2$ $x^4-y^4=(x^2+y^2)(x^2-y^2)=(x^2+y^2)(x+y)(x-y)$ ✓.

induction step: assume $p(n)$: $(x+y)\mid x^{2n}\ y^{2n}$

consider $(x^{2n}-y^{2n})(x^2+y^2)=x^{2n+2}-y^{2n+2}-x^2y^{2n}+y^2x^{2n}$

so $x^{2n+2}-y^{2n+2}=\underbrace{(x^{2n}-y^{2n})(x^2+y^2)}_{\text{divisible by } x+y \text{ by } p(n)}+\underbrace{x^2y^2(y^{2n-2}-x^{2n-2})}_{\text{divisible by } x+y \text{ by } p(n-1)}$ □.

Q18 **Thm** $1-\frac{1}{2}+\frac{1}{3}\ -\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$

Proof (induction) base case $n=1$: $1-\frac{1}{2}=\frac{1}{2}$ ✓.

induction step: assume true for $p(n)$ consider $1-\frac{1}{2}+\frac{1}{3}-\dots-\frac{1}{2n}+\frac{1}{2n+1}-\frac{1}{2n+2}$.

$=\frac{1}{n+1}+\dots+\frac{1}{2n}$ by $p(n)$.

$=\frac{1}{n+1}+\dots+\frac{1}{2n}+\frac{1}{2n+1}-\frac{1}{2n+2}$ $\quad \frac{1}{n+1}-\frac{1}{2(n+1)}=\frac{1}{2n+2}$

$=\frac{1}{n+2}+\dots+\frac{1}{2n+1}+\frac{1}{2n+2}$, giving $p(n+1)$ □

Q17 **Thm** $\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}<1$

Proof do Q18 induction, get $\frac{1}{n+1}+\dots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}\ -\frac{1}{2n}$

$=1+\underbrace{\left(-\frac{1}{2}+\frac{1}{3}\right)}_{<0}+\underbrace{\left(-\frac{1}{4}+\frac{1}{5}\right)}_{<0}+\dots+\underbrace{\left(-\frac{1}{2n-2}+\frac{1}{2n-1}\right)}_{<0}\underbrace{-\frac{1}{2n}}_{<0}<1$ □.

Q19 $x,y\in\mathbb{R}$, $x\sim y$ if $x-y\in\mathbb{Q}$

reflexive ~~symmetric~~: $x-x=0\in\mathbb{Q}$ so $x\sim x$.

symmetric $x-y\in\mathbb{Q}\Rightarrow y-x\in\mathbb{Q}$ so $x\sim y\Rightarrow y\sim x$

transitive: $x-y\in\mathbb{Q},\ y-z\in\mathbb{Q}\Rightarrow x-y+y-z=x-z\in\mathbb{Q}$, so $x\sim y,\ y\sim z\Rightarrow x\sim z$

yes: equivalence relation

Q20 $f:\mathbb{R}\to\mathbb{R}$, $f\sim g$ if $\exists A,B\in\mathbb{R}$ s.t. $f(x)<Ag(x)+B$

not symmetric: $f(x)=0,\ g(x)=x$, then $0\le 0x+1$ but $x\not\le B$ so $f\sim g$ but $g\not\sim f$.

Q21 $A \sim B$ if there is an injection $f: A \to B$.

not symmetric. $A = \{1\}$, $B = \{1,2\}$, injection $A \to B$ but not from $B \to A$.

Q22 $\{f : \mathbb{R} \to \mathbb{R}\}$ ~~(assume differentiable function~~

$f \sim g$ if $g = f'$ not reflexive: $f(x) = x$, $f'(x) = 1 \neq x$ so $f \not\sim f$