§1.3 Proof: An Introduction

A proof is a special kind of explanation ⇝ usually precise and formal
~~completeness~~ / level of detail depends on context
Q: why is the sum of two even numbers even?

Example

Thm 1.1 If a and b are even integers, then $a+b$ is an even integer.

Proof 1) begin by assuming a and b are even integers.
2) what does it mean to be even? A multiple of 2, i.e. $a = 2m$, $b = 2n$
for some other integers m, n.
3) what does this say about $a+b$? $a+b = 2m+2n = 2(m+n)$
sum of two integers is an integer, so $a+b$ is even. □.

Observations
- statement of theorem: a precise statement that can be proved, with specific assumptions. Typically if ... (hypothesis) then ... (conclusion) ← conditional sentence.

step 1: • assume hypothesis (and then derive conclusion)
• a, b known as variables: what do they mean?
where do they come from?
what are the rules for working with them?

step 2: interpret the assumption
in this case words ⇝ algebraic form.
— directly uses the definitions in this case (common in basic proofs).

remark a even means "a can be divided evenly by 2".
but also $a = 2m$ ← this turns out to be more useful.

step 3: do work, in this case
uses distributive property $2m+2n = 2(m+n)$.
— want intuition for this step, can be long and complicated.

GTHD 1: If x is an even integer, and y is a multiple of x, then y is even.

Conditional sentences   If .... then ....
hypothesis   conclusion.

Examples   If x, y odd integers, then xy odd

If two sides of a triangle have equal length, then two angles are equal

If a, b are real numbers, then $|a+b| \leq |a| + |b|$

If a function is differentiable, then it is continuous.

If P then Q. $(P \Rightarrow Q)$

doesn't matter what happens if P is false!!

Warning: lots of different ways to say this: eg $a+b$ ~~$x+y$~~ is even whenever a and b are even.

Variables   Thm 1.1 says hyp. says a, b ~~x, y~~ are even integers.

Q: which ones? - we don't know.

How not to prove Thm 1.1: suppose $a = 8$ and $b = 12$. Then $a+b = 20$, even.

need to deal with all even integers, so a is some even integer, we call this a variable.

Open sentences: a sentence containing a variable.

Example
- $a > b$ a, b variables (numbers)
- $S \subseteq \mathbb{Z}$ S variable (set)
- $A \cap B = \phi$ A, B variable (sets)
- $y^2 - 5y + 6 = 0$ y is a variable (number)
- $z \in C \cup D$ z, C, D variables (element, sets)
- a is even   a variable (integer).

substitution: a is even.   4 is even T

-7 is even F

0 is even T

$\underbrace{}_{\text{truth values}}$

the collection of objects which make a given open sentence true is called its truth set.

x is even : need to know what x can refer to.
in this case $x \in \mathbb{Z}$.
truth set is $\{-4, -2, 0, 2, 4, \dots\}$.

Example If $\underbrace{x \text{ is prime}, x > 2}_{\text{hyp.}}$, then $\underbrace{x \text{ is odd}}_{\text{conclusion}}$.

truth set $\{3, 5, 7, 11, \dots\}$ $\{-1, 1, 3, 5, 7, 9, \dots\}$.

When is a conditional statement true?
when truth set for hypothesis is a subset of truth set for conclusion.
When is a conditional statement false?
" not a subset "

Counterexamples: Example: if x is a positive integer then x is even.
$\{1, 2, 3, 4, \dots\}$ $\{\dots, -2, 0, 2, 4, \dots\}$

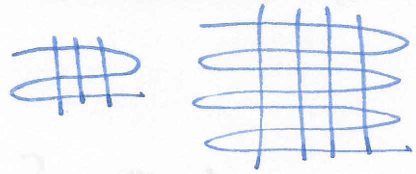
just need to find 1 example which doesn't work, e.g. 3.

Note: a conditional sentence is false if values can be found for the variables which make the hyp. true and the conclusion false.

Note: substitutions: hyp true conclusion true ✓.
hyp false true ✓
false false ✓.
true false X counterexample.

Definitions Example: Defn An even integer is divisible by 2. (if and only if!)
not a definition: an integer which is divisible by 4 is even. (not iff).

Motivation: proof for certainty
understanding.



Q: how many pieces?