Math 505 Introduction to Proofs Spring 18 Sample Midterm 2

(1) Show that $\sqrt[3]{5}$ is irrational.

(2) Find an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ which is
  (a) injective but not surjective
  (b) surjective but not injective

(3) Find an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is
  (a) injective but not surjective
  (b) surjective but not injective

(4) Find explicit bijections between the following subintervals of $\mathbb{R}$.
  (a) $(0,1)$ and $(1,\infty)$
  (b) $(1,\infty)$ and $(0,\infty)$
  (c) $(0,\infty)$ and $\mathbb{R}$
  (d) $(0,1)$ and $\mathbb{R}$

(5) Say a set A is *countable* if there is an injective map $f\colon A \to \mathbb{N}$. Show that the product of countable sets is countable. Show that the set of solutions of linear equations $ax + b = 0$, with $a, b \in \mathbb{Z}$ is countable. What about solutions of quadratic equations $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{Z}$?

(6) Either give examples of functions with the properties below, or explain why they don't exist.
  (a) An injective function from $\mathbb{Z}$ to $\mathbb{R}$
  (b) A surjective function from $\mathbb{R}$ to $\mathbb{Z}$
  (c) A surjective function from $\mathbb{Z}$ to $\mathbb{R}$
  (d) An injective function from $\mathbb{R}$ to $\mathbb{R}^2$
  (e) A surjective function from $\mathbb{R}^2$ to $\mathbb{R}$
  (f) A bijective function from the unit interval $[0,1]$ to the unit square $[0,1] \times [0,1]$.

(7) State the negation of the following statements, using appropriate quantifiers.
  (a) e^π is rational.
  (b) The function f is surjective and injective.
  (c) The integer n is divisible by 2 or 3.
  (d) There is an injective function $f\colon A \to B$.
  (e) The function $f\colon \mathbb{R} \to \mathbb{R}$ is bounded above.
  (f) $\lim_{x \to 0} f(x) = 0$.

(8) Write out careful proofs, or give counterexamples, to the following statements.
   (a) Show that the sum of any two adjacent integers is odd.
   (b) Show that the sum of any three consecutive integers is divisible by 3.
   (c) $f\colon A \to B$ has an inverse if and only if it is both surjective and injective.
   (d) If f is injective, then $f \circ g$ is injective.
   (e) If f and $f \circ g$ are surjective, then g is surjective.
   (f) If $f\colon \mathbb{R} \to \mathbb{R}$ is increasing, and $g\colon \mathbb{R} \to \mathbb{R}$ is increasing, then $f \circ g$ is increasing.
   (g) If $x \in \mathbb{R}$ and $x^2 \leq x$, then $x \leq 1$.

(9) Suppose that $f\colon A \to B$ and let $C \subseteq A$.
   (a) Prove or give a counterexample: $f(A - C) \subseteq f(A) - f(C)$
   (b) Prove or give a counterexample: $f(A) - f(C) \subseteq f(A - C)$
   (c) What condition on f will ensure that $f(A - C) = f(A) - f(C)$? Prove this.
   (d) What condition on f will ensure that $f(A - C) = B - f(C)$? Prove this.