Math 505 Introduction to Proofs Spring 18 Sample Final

(1) Consider the statement

If x and y are real numbers with $x < y$ then $x^2 < y^2$.

Which, if any, of the following substitions gives a counterexample:

(a) $x = 1, y = 2$ (b) $x = 2, y = -1$ (c) $x = -2, y = -1$

(2) State which of the following statements are true, vacuously true, or false.

(a) $(A \cap B)' = A' \cup B'$

(b) If $\mathcal{P}(A) = \varnothing$, then $A = \varnothing$

(c) $(A \cup B) - C = (A - C) \cap (B - C)$

(3) Let $f \colon X \to Y$ be a function, and let A and B be subsets of X. Show that $f(A \cup B) = f(A) \cup f(B)$.

(4) Let $f \colon X \to Y$ be a function, and let A be a subset of Y. Show that $f(f^{-1}(A)) \subset A$. Is $f(f^{-1}(A)) = A$?

(5) Either give examples of functions with the properties below, or explain why they don't exist.

(a) A surjective function from $[0, 1]$ to $\mathbb{N}$

(b) A surjective function from $(0, 1)$ to $(0, \infty)$

(c) An injective function from $\mathbb{R}^2$ to $\mathbb{R}$

(6) State the negation of the following statements, using appropriate quantifiers.

(a) The function $f \colon A \to B$ is surjective.

(b) There is a surjective function $f \colon A \to B$.

(c) The function $f \colon \mathbb{R} \to \mathbb{R}$ is increasing.

(7) Write out careful proofs, or give counterexamples, to the following statements.

(a) If n is an integer then $n^2 + 3n$ is even.

(b) If f and $g \circ f$ are injective, then g is injective.

(c) If $f \colon \mathbb{R} \to \mathbb{R}$ is decreasing, and $g \colon \mathbb{R} \to \mathbb{R}$ is decreasing, then $f + g$ is decreasing.

(8) Write out the converse to the statement "If $p(x)$ and $q(x)$ then $r(x)$".

(9) Write out the contrapositive to the statement "If $p(x)$ and $q(x)$ then $r(x)$".

(10) We say a sequence a_n has a limit L if for all $\epsilon > 0$ there is an N such that $|L - a_n| \leq \epsilon$ for all $n \geq N$. Write this statement out using the quantifier

symbols, then write out the negation of this statement. Use the negation to show that the sequence $a_n = (-1)^n$ does not have a limit.

(11) Show that the following numbers are irrational: $\sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{2} + \sqrt{3}$.

(12) Show that $1 + 4 + 7 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 2)$.

(13) Show that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all $n \geq 0$.

(14) Let F_k be the k-th Fibonacci number. Show that $F_1F_2 + F_2F_3 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$.

(15) Show that $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$.

(16) Consider the relation on $\mathbb{R}$ determined by $x \sim y$ if $x - y$ is rational. Is this an equivalence relation?

(17) Consider the relation on functions $f \colon \mathbb{R} \to \mathbb{R}$ given by $f \sim g$ if there is a number $c \in \mathbb{R}$ such that $f(x) = g(x) + c$. Is this an equivalence relation?

(18) Define a relation on sets by $A \sim B$ if there is a bijection between A and B. Is this an equivalence relation on sets? What are the equivalence classes?

(19) Let F be the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$. Define a relation on F by $f \sim g$ if $g = f'$. Is this an equivalence relation? Does it define a function?