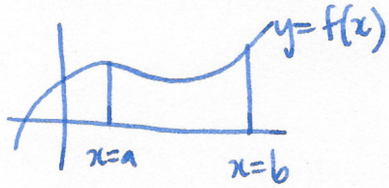
why: $c(t) = (x(t), y(t))$ chain rule: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

Example $c(t) = (\cos t, \sin 2t)$

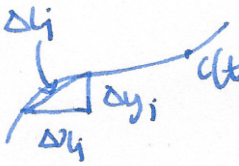
$$\left.\begin{aligned} x(t) &= \cos t & y(t) &= \sin 2t \\ \frac{dx}{dt} &= -\sin t & \frac{dy}{dt} &= 2\cos 2t \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{2\cos 2t}{-\sin t}$$

§11.2 Arc length and speed

recall



arc length $= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

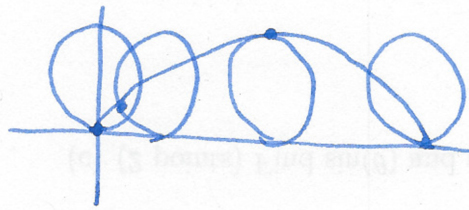


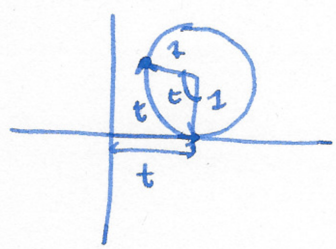
$$\text{arc length} = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example cycloid





$$\begin{aligned} x(t) &= t - \sin t & \frac{dx}{dt} &= 1 - \cos t \\ y(t) &= 1 - \cos t & \frac{dy}{dx} &= \sin t \end{aligned}$$

arc length: $\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$ recall: $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

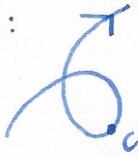
$= 1 - 2\sin^2\theta$

$= \int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\, dt$ $2\sin^2\theta = 1 - \cos 2\theta$

$2\sin^2\left(\frac{t}{2}\right) = 1 - \cos t$

$= \int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt = 4\int_0^{\pi} \sin\frac{t}{2}\, dt = 4\left[-2\cos\frac{t}{2}\right]_0^{\pi} = 8(0+1) = 8$

speed:



$c(t) = (x(t), y(t))$

$$\text{speed} = \frac{d}{dt}(\text{arc length}) = \frac{d}{dt}\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

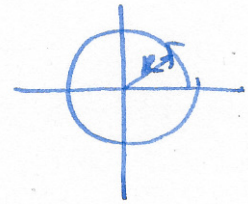
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \text{length of velocity vector.}$$

Example ① $c(t) = (t, t^2)$ $x(t) = t$ $y(t) = t^2$ $\frac{dx}{dt} = 1$ $\frac{dy}{dt} = 2t$ speed $s'(t) = \sqrt{1+4t^2}$

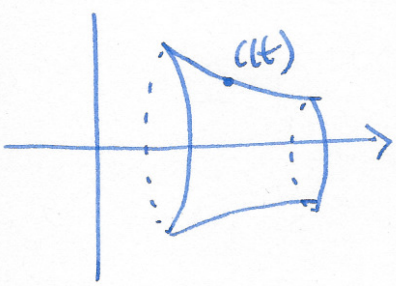
Example ② $c(t) = (R\cos\omega t, R\sin\omega t)$

$c'(t) = (-\omega R\sin\omega t, \omega R\cos\omega t)$

$\|c'(t)\| = \sqrt{R^2\omega^2\sin^2\omega t + \omega^2 R^2\cos^2\omega t} = R|\omega|$.



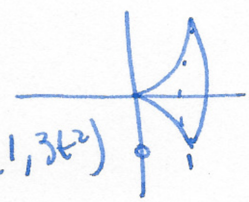
surface area for parametrized curves



$$\text{surface area} = 2\pi\int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

example $c(t) = (t, t^3)$

$c'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1, 3t^2)$



$$\text{surface area} = 2\pi\int_0^1 t^3\sqrt{1^2 + (3t^2)^2}\,dt = 2\pi\int_0^1 t^3\sqrt{1+9t^4}\,dt$$

$u = 1+9t^4$

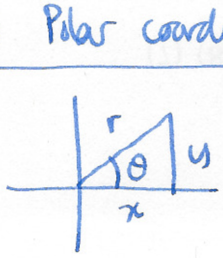
$\frac{dy}{dt} = 36t^3$

$$2\pi\int_0^{10} t^3\sqrt{u}\cdot\frac{dt}{du}\,du = 2\pi\int_1^{10} t^3\sqrt{u}\cdot\frac{1}{36t^3}\,du = \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du$$

$$= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\pi}{27}\left(10^{3/2} - 1\right) \approx 3.56$$

Example $c(t) = (t - \tanh t, \operatorname{sech} t)$

§11.3 Polar coordinates



$x = r\cos\theta$
$y = r\sin\theta$

$\tan\theta = y/x$
$r^2 = x^2 + y^2$