<u>Thm</u> If $f(x)$ is equal to a power series centered at $x=c$, with radius of convergence $R>0$, then $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, where $a_n = \frac{f^{(n)}(c)}{n!}$

<u>Useful facts</u> if the Taylor series converges, we can:

- differentiate
- integrate
- multiply (xe^x)
- substitute (e^{x^2})

<u>Examples</u> xe^x, e^{-x^2}, $e^x \sin x$

<u>Fact</u>: works for complex numbers $x+iy$

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \\ &\quad + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

How to make l'Hôpital's rule problems:

$$\left.\begin{aligned} \sin(2x) &\approx 2x - \frac{(2x)^3}{3!} + o(x^5) \\ e^{3x} &\approx 1 + 3x + o(x^2) \end{aligned}\right\} \quad \frac{\sin(2x)}{e^{3x}-1} \approx \frac{2x}{3x} = \frac{2}{3}.$$

Cheap Error bounds for e^x: $(x<1)$

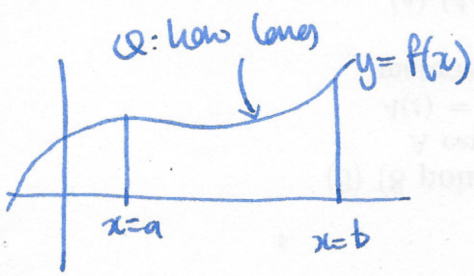
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{x^n}{n!} + \underbrace{\frac{x^{n+1}}{(n+1)!} + \cdots}$$

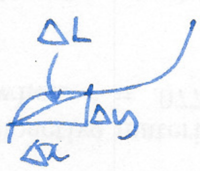
$$\leq x^{n+1} + x^{n+2} + \cdots = \frac{x^{n+1}}{1-x}.$$

so $e^{1/2} = 1 + 1/2 + \frac{(1/2)^2}{2!} + \cdots + \frac{(1/2)^n}{n!}$ with error $\leq \frac{1}{2^{n+2}}$.

§8.1 Arc length and surface area



A: ΔL

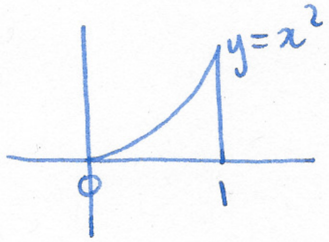


$$|\Delta L| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

length $= \sum_{i=1}^{n} |\Delta L_i| = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\, \Delta x_i$

take limit as $|\Delta x_i| \to 0$: arc length $= \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Example ①



$\frac{dy}{dx} = 2x$ arc length $\int_0^1 \sqrt{1 + 4x^2}\, dx$

(trig sub integral)

② $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1,2]$

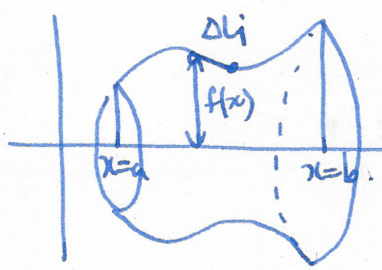
$f'(x) = \frac{1}{4}x^2 - \frac{1}{x^2}$

arc length $= \int_1^2 \sqrt{1 + \left(\frac{1}{4}x^2 - \frac{1}{x^2}\right)}\, dx = \int_1^2 \sqrt{1 + \frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4}}\, dx$

$= \int_1^2 \sqrt{\frac{1}{16}x^4 + \frac{1}{2} + \frac{1}{x^4}}\, dx = \int_1^2 \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2}\, dx = \int_1^2 \frac{x^2}{4} + \frac{1}{x^2}\, dx$

$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^2 = \frac{8}{12} - \frac{1}{4} - \frac{1}{12} + 1 = 1\frac{1}{3}$.

Area of surface of revolution

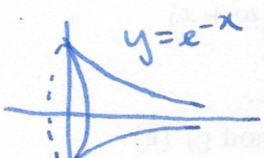


area $\approx \sum_{i=1}^{n} |\Delta L_i|\, f(x)\, 2\pi$ $\quad |\Delta L_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ etc.

take limit as $|\Delta x_i| \to 0$

surface area of sol of revolution $= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$

Example



$\frac{dy}{dx} = -e^{-x}$ surface area $= \int_0^\infty 2\pi e^{-x} \sqrt{1 + (-e^{-x})^2}\, dx$.

sub $u = e^{-x}$, $\frac{du}{dx} = -e^{-x}$ $= \int_0^\infty 2\pi \cdot u \sqrt{1 + u^2}\, \frac{dx}{du}\, du$