<u>Example</u> show $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ converges (for large n, $a_n \sim \frac{1}{n^2}$)

compare with $b_n = \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^2}{n^4-n-1}}{\frac{1}{n^2}} = \frac{n^4}{n^4-n-1} = 1.$

so $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{n^4-n-1}$ converges, by limit comparison test.

<u>Example</u> does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$ converge? compare with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-9}}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-9}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-9/n^2}} = 1$$

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-9}}$ diverges, by limit comparison test.

§10.4 Absolute and conditional convergence

Q: what about $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$? (*)

<u>Defn</u> A series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

(*) is absolutely convergent.

<u>Example</u> $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ <u>not</u> absolutely convergent.

<u>Thm</u> Absolute convergence $\Rightarrow$ convergence.

<u>Proof</u> $0 \leq a_n + |a_n| \leq 2|a_n|$

$\sum_{n=1}^{\infty} 2|a_n| = \underbrace{2\sum_{n=1}^{\infty} |a_n|}_{\text{converges}} \Rightarrow \sum_{n=1}^{\infty} a_n + |a_n|$ converges, by comparison test.

then $\sum_{n=1}^{\infty} a_n + |a_n| - |a_n|$ ~~converges~~ $= \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$

converges. $\Leftarrow$ converges converges

$= \sum_{n=1}^{\infty} a_n$, so this converges. $\square$.

Q: what about $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

Def

<u>Def^n</u> $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u> if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

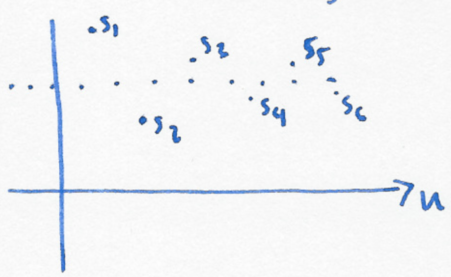
<u>Thm</u> (Alternating series test) Let a_n be a positive, decreasing sequence, $a_n \to 0$. then $\underbrace{\sum_{n=1}^{\infty} (-1)^n a_n}_{s}$ converges. Furthermore $0 \leq s \leq a_1$ and $s_{2n} \leq s \leq s_{2n+1}$ for all n.

<u>Proof</u>: even partial sums: $s_{2n} = \underbrace{a_1 - a_2}_{\geq 0} + \underbrace{a_3 - a_4}_{\geq 0} + \cdots + \underbrace{a_{2n-1} - a_{2n}}_{\geq 0}$

positive increasing sequence.

odd partial sums: $s_{2n+1} = a_1 - \underbrace{(a_2 - a_3)}_{\geq 0} - \underbrace{(a_4 - a_5)}_{\geq 0} - \cdots - \underbrace{(a_{2n} - a_{2n+1})}_{\geq 0}$

decreasing sequence



furthermore $s_{2n} = a_1 - (a_2 - a_3) - \cdots - a_{2n}$

so $s_{2n} \leq a_1$ for all n

so s_{2n} is an increasing sequence, bounded above

so $\lim_{n\to\infty} s_{2n}$ exists.

similarly $\lim_{n\to\infty} s_{2n+1}$ exists.

note: $\lim_{n\to\infty} s_{2n} - s_{2n+1} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} -a_{2n+1} = 0$ $\square$.

Example show $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$ converges (alternating harmonic series)

use alternating series test. $a_n = \frac{1}{n}$

a_n: positive, decreasing, $a_n \to 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. $\square$.

so $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$ is conditionally convergent.

§10.5 Ratio and root tests

fact $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ Q: how do we show this converges?

(e.g: use comparison test $n! = 1.2.3 \ldots (n-2)(n-1)n > (n-1)^2$ so $\frac{1}{n!} < \frac{1}{(n-1)^2}$)

Thm Ratio test (a_n) sequence, and suppose $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

then ① if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

② if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

③ if $\rho = 1$ no information.

Proof if $\rho < 1$, there is a number $\rho < r < 1$, and a number N s.t.

$\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq N$, so $|a_{N+1}| < r|a_N|$

$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$ etc.

so $\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| r^n \leq \frac{|a_N|}{1-r}$ so converges by comparison test with geometric series.

if $\rho > 1$, then there is $\rho > r > 1$ and N s.t.

$\frac{|a_{N+1}|}{|a_N|} > r$ for all $n \geq N$, so $a_n \not\to 0 \Rightarrow$ diverges. $\square$.

Example ① show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges

ratio test: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$.

② show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$

Bad example $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2 = 1$

Thm Root test (a_n) sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists

① if $L<1$ then $\sum a_n$ converges absolutely

② if $L>1$ then $\sum a_n$ diverges

③ if $L=1$ no information

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

§10.6 Power series

Defn A power series centered at $x=a$ is an infinite sum of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

note: this gives a function of x if the series converges.

the series always converges for $x=a$! $F(a) = a_0$.

Thm Radius of convergence Let $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$, then

① $F(x)$ converges only for $x=a$ $(R=0)$

or ② $F(x)$ converges for all x $(R=\infty)$

or ③ there is an $R>0$ s.t. $F(x)$ converges for all $|x-a|<R$

and diverges for all $|x-a|>R$. It may or may not converge at $a+R$, $a-R$.

R is called the radius of convergence

Example for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{2} = \frac{|x|}{2}$