$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

$$s_1 = 1 - \frac{1}{2}$$

$$s_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}.$$

$$s_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

so $s_N = 1 - \frac{1}{N+1}$ $\quad \lim_{N\to\infty} s_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$

Example (harmonic series) $\underbrace{1}_{\geq 1} + \underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}}$

diverges!

Q: when does a series converge? **A**: dunno.

Tools:

- **Thm** **Divergence test** if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ **diverges**.

Warning: $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Recall $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

Rules for series, AKA infinite sums

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series.

then
$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

observation: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^{2n}}$

§10.3 Positive series

positive series : $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ all $a_n \geq 0$

<u>note</u>: the sequence s_N is an <u>increasing</u> sequence : $s_{N+1} = s_N + \underbrace{a_{n+1}}_{\geq 0}$

<u>recall</u>: an <u>increasing sequence</u> with an <u>upper bound</u> converges.

<u>Thm</u> Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs:

① the partial sums are bounded above, and S converges

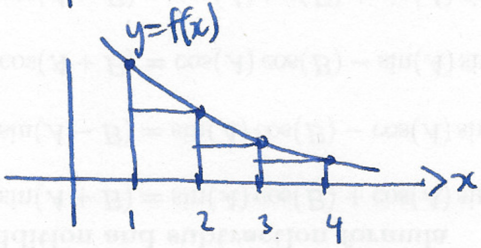
② the partial sums are not bounded above, and S diverges.

<u>Thm</u> <u>Integral test</u> Let $a_n = f(n)$ $\quad f(x)$ $\left.\begin{matrix}\text{positive}\\ \text{decreasing}\\ \text{continuous}\end{matrix}\right\}$ for $x \geq 1$

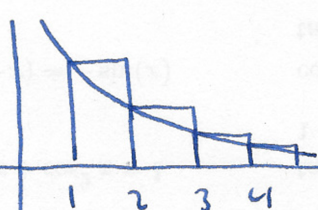
then ① if $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

② if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Proof</u>



$$s_N - a_1 = a_2 + a_3 + \cdots + a_N \leq \int_1^N f(x)\,dx$$



$$s_N = a_1 + a_2 + \cdots + a_N \geq \int_1^N f(x)\,dx \qquad \square.$$

<u>Example</u> $S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \qquad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$\int_1^{\infty} x^{-1/2}\,dx = \lim_{N\to\infty} \int_1^N x^{-1/2}\,dx = \lim_{N\to\infty} \left[2x^{1/2}\right]_1^N = \lim_{N\to\infty} 2\sqrt{N} - 2 \to \infty \text{ as } N \to \infty.$$

so S diverges.

Thm Convergence of p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$

Proof (integral test) $\sum_{n=1}^{\infty} a_n$ $\quad a_n = f(n)$ $\quad f(x) = \frac{1}{x^p} = x^{-p}$

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$, diverges if $p \le 1$ $\square$.

Example: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$)

$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (to ?).

Thm Comparison test: suppose $0 \le a_n \le b_n$ for all $n \ge M$.

then ① if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

② if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example $\sum_{n=1}^{\infty} 2^{-n^2} = \frac{1}{2}+\frac{1}{2^4}+\frac{1}{2^9}+\cdots$

note: $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ geometric series, converges.

Thm Limit comparison test: a_n, b_n positive sequences.

suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$ exists

then
- if $L>0$: $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges
- if $L=0$: $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof (sketch) case $L>0$, then $\frac{a_n}{b_n} \to L$, so $0 < \frac{a_n}{b_n} < R$ $\quad L<R$

$0 < a_n < Rb_n$

comparison test: $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

similarly $\frac{b_n}{a_n} \to \frac{1}{L}$, so $0 < \frac{b_n}{a_n} < R'$ for some $\frac{1}{L} < R'$ $\quad 0 < b_n < R' a_n$

comparison test $\sum a_n$ converges $\Rightarrow \sum b_n$ converges $\square$

(If $L=0$ only get one direction)