Thm. If $f(x)$ is cts and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$.

Important: f continuous (at L) Bad example $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$

then $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) = 1$ for all n and $f(0) = 0 \neq \lim_{n\to\infty} f(\frac{1}{n}) = 1$

Example find $\lim_{n\to\infty} e^{n/n+1}$

start with $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$, then $\lim_{n\to\infty} e^{n/n+1} = e^{\lim_{n\to\infty} n/n+1} = e^1 = e$

Defn A sequence (a_n) is
- bounded above if $a_n \le M$ for all n
- bounded below if $L \le a_n$ for all n
- bounded if $L \le a_n \le M$ for all n.

Thm Convergent subsequence are bounded

Warning: bounded subsequences need not converge.

Example: $0,1,0,1,0,1,\dots$ $a_n = \frac{1-(-1)^n}{2}$

Thm Bounded monotonic subsequences converge.
- if (a_n) is increasing and $a_n \le M$ then $a_n \to l \le M$
- if (a_n) is decreasing and $L \le a_n$ then $a_n \to l \ge L$

Example $a_n = \frac{1}{n}$ show decreasing, want $a_n \ge a_{n+1}$

$n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$ lower bound $L = -100$ $\lim_{n\to\infty} \frac{1}{n} = l \ge -100$.

Example show $a_n = \sqrt{n+1} - \sqrt{n}$ decreasing and bounded below.

note $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$ as $\sqrt{x}$ monotonic $\Rightarrow a_n \ge 0$

so can choose lower bound $L = 0$.

decreasing: consider $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - (x)^{1/2}$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$$

claim: $f'(x) < 0$:

$$x+1 > x$$
$$\sqrt{x+1} > \sqrt{x}$$
$$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$$ so $f'(x) < 0 \Rightarrow f(x)$ decreasing. □.

Alternately: $\frac{\sqrt{n+1} - \sqrt{n}}{} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. decreasing.

§10.2 Series

Defn A series is an infinite sum $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

Examples $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ $\qquad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$1 + 1 + 1 + 1 + \cdots$ $\qquad 1 - 1 + 1 - 1 + 1 - \cdots$

Defn The N-th partial sum $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

Defn The sum of the infinite series is defined to be the limit of partial sums, if this limit exists.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_N$$

If $\lim_{N\to\infty} S_N = s$, then we say $\sum_{n=1}^{\infty} a_n$ converges and write $\sum_{n=1}^{\infty} a_n = s$

Example ① $1+1+1+\cdots$ $\qquad S_N = \underbrace{1+1+\cdots+1}_{N} = N \qquad \lim_{N\to\infty} N = \infty$

so $\sum_{n=1}^{\infty} 1$ does not converge.

② $1-1+1-1+1-1\cdots$ $\qquad s_1 = \cancel{0}1,\ s_2 = 0,\ s_3 = 1\ \ s_4 = 0 \ \cdots$

$(s_n) = 1, 0, 1, 0, 1, \ldots$ does not converge.

Warning: can't re-arrange non-converging sums: $(1-1)+(1-1)+(1-1)+\cdots = 0$

$1 + (-1+1) + (-1+1) + \cdots = 1$.

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty}\frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4}$$

$$s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8}$$

$$S_N = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^N}$$

$$\frac{1}{2}S_N = \frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^N}+\frac{1}{2^{N+1}}$$

$$S_N-\frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$\frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$S_N = 1-\frac{1}{2^N} \qquad \lim_{N\to\infty} 1-\frac{1}{2^N} = 1, \text{ so } \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1$$

General case

$$c+cr+cr^2+cr^3+\cdots = \sum_{n=0}^{\infty} cr^n$$

$$S_N = c+cr+cr^2+\cdots+cr^N$$

$$rS_N = cr+cr^2+\cdots+cr^N+cr^{N+1}$$

$$S_N-rS_N = c-cr^{N+1}$$

$$S_N(1-r) = c(1-r^{N+1})$$

$$S_N = c\,\frac{1-r^{N+1}}{1-r} \qquad \lim_{N\to\infty} c\,\frac{1-r^{N+1}}{1-r} = \frac{c}{1-r}, \text{ if } |r|<1$$

otherwise does not converge.

Special series (Telescoping)

Example $$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$$