§10.1 sequences

Def^n A sequence is a list of numbers indexed by $\mathbb{N}$ = positive integers.

Example $1, 2, 3, 4, 5, 6, \dots$ Notation: $a_1, a_2, a_3, \dots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$ or $(a_n)_{n \in \mathbb{N}}$

$1, 27, \pi, \sqrt{2}, \dots$ where a_n is the n-th number in the sequence.

$1, 1, 1, 1, 1, \dots$

Q: what is not a sequence? a number, a set of numbers, a function...

sometimes (but not always) we can give the sequence by a formula.

Examples $(a_n)_{n \in \mathbb{N}}$ $a_n = n$ $(n)_{n \in \mathbb{N}} = (1, 2, 3, 4, \dots)$

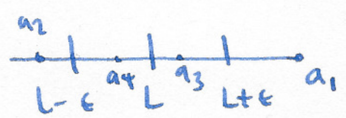
$(a_n)_{n \in \mathbb{N}}$ $a_n = \frac{1}{1+n^2}$ $\left(\frac{1}{1+n^2}\right)_{n \in \mathbb{N}} = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \dots\right)$

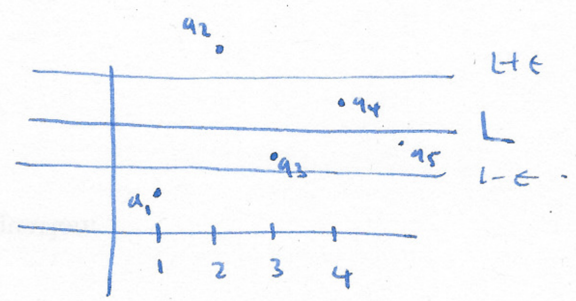
Example (recursive def^n)

$a_{n+2} = a_{n+1} + a_n$ $a_1 = 1$ $a_2 = 1$ gives $1, 1, 2, 3, 5, 8, 13, \dots$ (Fibonacci sequence)

Def^n A sequence (a_n) converges to L if for every $\epsilon > 0$ there is an N st

$|a_n - L| \leq \epsilon$ for all $n \geq N$





Notation $\lim_{a_n \to L} a_n = L$ or $a_n \to L$

Examples $(a_n) = \left(\frac{1}{n}\right)$ $\lim_{n \to \infty} \frac{1}{n} = 0$

Proof given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$ if $n > N$ then

$\frac{1}{n} < \frac{1}{N} < \epsilon$, as required. $\square$.

special case: sequence defined by a function $f(x)$, i.e. $a_n = f(n)$

Thm If $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} f(n) = L$

Q: is the converse true?

Example $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ $a_n = \frac{n-1}{n}$ $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n \to \infty} a_n = 1$

Example (geometric sequences) $a_n = r^n$

e.g. $2, 4, 6, 8, 16, \ldots$ $\quad a_n = 2^n$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $\quad a_n = \frac{1}{3^n}$

$1, 1, 1, 1, \ldots$ $\quad a_n = 1^n$

Fact: $\lim_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \\ \text{DNE} & r \leq -1 \end{cases}$

Rules for limits of sequences: same as rules for limits of functions

suppose $a_n \to L$ and $b_n \to M$, then

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$
- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$ $\quad (M \neq 0)!$
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$ $\quad$ (c constant, i.e. does not depend on n)

Squeeze Thm If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

Example $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any R.

Proof there is an integer M s.t. $M \leq R \leq M+1$

$$0 \leq \frac{R^n}{n!} = \underbrace{\frac{R}{1} \cdot \frac{R}{2} \cdots \frac{R}{M}}_{\text{call this } A} \, \underbrace{\frac{R}{M+1} \cdots \frac{R}{n-1}}_{\leq 1} \cdot \frac{R}{n} \leq A \frac{R}{n}$$

so $0 \leq \frac{R^n}{n!} \leq A \cdot \frac{R}{n}$ $\quad \lim_{n\to\infty} 0 = 0 \quad \lim_{n\to\infty} A\frac{R}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{R^n}{n!} = 0$ $\square$.