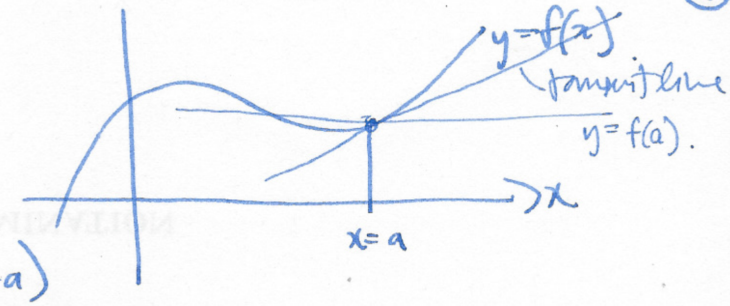
§8.4 Taylor Polynomials

Approximating functions at $x=a$:



0-th approx: $f(x) \approx f(a)$

1-st approx: (straight line) $f(x) \approx f(a) + f'(a)(x-a)$

2-nd approx: (quadratic)

3-rd approx: (cubic) ? how do we find these?

tangent line: $y - y_0 = m(x - x_0)$

$y - f(a) = f'(a)(x-a)$

tangent line: unique line with same value as $f(a)$ at $x=a$ $\quad y = mx + c$.
same tangent at $x=a$ $f'(a)$.

quadratic: same value at $x=a$: $f(a)$. $\quad y = \cancel{ax^2} \; ax^2 + bx + c$
same tangent / first derivative at $x=a$: $f'(a)$
same 2nd derivative at $x=a$ $f''(a)$

cubic: same value and first 3 derivatives at $x=a$. $f(a), f'(a), f''(a), f^{(3)}(a)$.

etc.

<u>quadratic</u> $T_2(x) = ax^2 + bx + c$

better: $T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$

deg 0: $T_2(a) = a_0 \qquad = f(a)$ (same value)

deg 1: $T_2'(x) = a_1 + 2a_2(x-a)$

$T_2'(a) = a_1 \qquad = f'(a)$ (same derivative)

deg 2: $T_2''(x) = 2a_2$

$T_2''(a) = 2a_2 \qquad = f''(a)$ (same 2nd derivative)

so $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

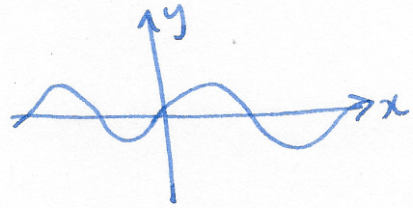
<u>in general</u> $T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$

differentiate k times: $T_n^{(k)}(x) = k!\,a_k + $ (stuff with $(x-a)$ factors)

$$T_n^{(k)}(a) = k!\,a_k = f^{(k)}(a) \qquad \text{so } a_k = \frac{1}{k!} f^{(k)}(a)$$

so $$T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$



Example ① $y = \sin x$ at $x = 0$

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f^{(3)}(x) = -\cos(x)$ $\quad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$

so $$T_4(x) = 0 + 1.x + \frac{0}{2!}.x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}.x^4$$
$$= x - \frac{x^3}{3!}.$$

② $y = \cos(x)$ at $x = 0$

$f(x) = \cos(x)$ $\quad f(0) = 1$

$f'(x) = -\sin(x)$ $\quad f'(0) = 0$

$f''(x) = -\cos(x)$ $\quad f''(0) = -1$

$f^{(3)}(x) = \sin(x)$ $\quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x)$ $\quad f^{(4)}(0) = 1$

so $$T_4(x) = 1 + 0x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4$$
$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

③ $y = e^x$ at $x = 0$

$f(x) = e^x$ $\quad f(0) = \phi$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$$T_2(x) = 1 + x + \frac{x^2}{2!}$$

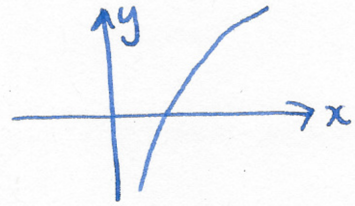
Sneak preview : what about infinite sums? $\quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

↑ not polynomial.
called a Taylor series.

Fact : $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Remark $$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = \begin{matrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \\ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \end{matrix} = \cos\theta + i\sin\theta.$$

Example $y = \ln(x)$ at $x = 1$



$f(x) = \ln(x)$ $\quad f(1) = 0$

$f'(x) = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $\quad f^{(3)}(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(1) = 4!$

$f^{(5)}(x) = 4!\,x^{-5}$

$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,x^{-k}$ $\quad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

so $T_n(x) = 0 + 1(x-1) + (-1)\frac{(x-1)^2}{2!} + \frac{2}{3!}(x-1)^3 + \cdots (-1)^{n+1}\frac{(n-1)!}{n!}(x-1)^n.$

Taylor series is $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

__warning__: this may not converge!

$T(3) = 2 - \frac{2^2}{2} + \frac{3^2}{3} - \frac{2^4}{4} + \cdots$

__Q__: how good are the Taylor polynomial approximations?

__Thm__ (error bound) $f(x)$ function, $f^{(n)}(x)$ exists and is cts.

Let K be an upper bound on $|f^{(n+1)}(u)|$ for all u in $[a,x]$

then $|T_n(x) - f(x)| \leq \frac{K|x-a|^{n+1}}{(n+1)!}$

Example $f(x) = e^x$ find $T_4(x)$ at $x=0$, find an error bound for $T_4(1)$

$f(x) = e^x$ $\quad f(0) = 1$ $\qquad T_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$

$\vdots$

$f^{(4)}(x) = e^x$ $\quad f^{(4)}(0) = 1$ $\qquad T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.708\overline{33}.$

__error bound__: need to find K s.t. $|f^{(n+1)}(x)| \leq K$ for all $x \in [0,1]$

i.e. want $|e^u| \leq K$ for $u \in [0,1]$, can choose $K = 3$.

so $|e - T_4(1)| \leq \frac{\cancel{e}3|1|^{n+1}}{(n+1)!} = \frac{3}{120}$

quick upper bound for e:

$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

$\leq 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3.$