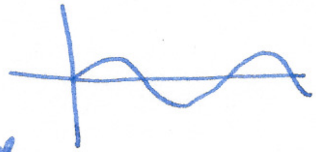
warning: sometimes the limit doesn't exist.

Example $\int_0^\infty \sin(x)\,dx$

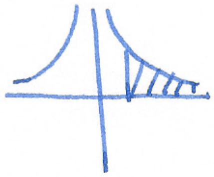


$$\int_0^R \sin(x)\,dx = \left[-\cos(x)\right]_0^R = 1-\cos(R) \qquad \lim_{R\to\infty} 1-\cos(R) \quad \text{DNE ?}$$

e.g. $R = 2\pi n$: $1-\cos(2\pi n) = 0$

$2\pi n+\pi$: $1-\cos(2\pi n+\pi) = 2$

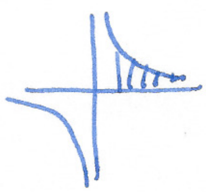
Example ① $\int_1^\infty \frac{1}{x^2}\,dx$



$$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = -\frac{1}{R}+1$$

$$\lim_{R\to\infty} 1-\frac{1}{R} = 1$$

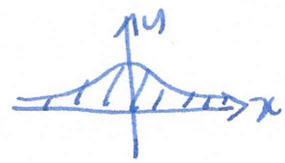
② $\int_1^\infty \frac{1}{x}\,dx$



$$= \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty}\left[\ln|x|\right]_1^R = \lim_{R\to\infty} \ln|R| - 0$$

$\to \infty$ as $R\to\infty$.

Doubly infinite integrals $\int_{-\infty}^{\infty} f(x)\,dx$ (f continuous!)



Def^{n} (f cts) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx$, provided each limit exists.

Example
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to\infty}\int_{-R}^{0}\frac{1}{1+x^2}\,dx + \lim_{R\to\infty}\int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\left[\tan^{-1}(x)\right]_{-R}^{0} + \lim_{R\to\infty}\left[\tan^{-1}(x)\right]_0^R$$

$$= \lim_{R\to\infty} 0 + \tan^{-1}(R) + \lim_{R\to\infty}\tan^{-1}(R) - 0 = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

warning
$$\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$$

Example $\int_{-\infty}^{\infty} \sin(x)\,dx$ DNE, but $\lim_{R\to\infty}\int_{-R}^{R}\sin(x)\,dx = \lim_{R\to\infty}\left[-\cos(x)\right]_{-R}^{R}$

$= \lim_{R\to\infty} -\cos(R)+\cos(-R) = \lim_{R\to\infty} 0 = 0.$

Example $\int_0^\infty xe^{-x}\,dx = \lim_{R\to\infty}\int_0^R xe^{-x}\,dx$ $\qquad \int uv'\,dx = uv - \int u'v\,dx$

$u = x \quad u' = 1$

$v' = e^{-x} \quad v = -e^{-x}$

$= \lim_{R\to\infty}\left[-xe^{-x}\right]_0^R - \lim_{R\to\infty}\int_0^R -e^{-x}\,dx$

$= \underbrace{\lim_{R\to\infty} -Re^{-R}}_{=0} + \lim_{R\to\infty}\left[-e^{-x}\right]_0^R = \lim_{R\to\infty} -e^{-R}+1 = 1$

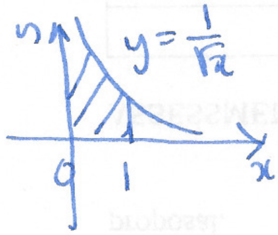
Example when does $\int_1^\infty \frac{1}{x^p}\,dx$ exist? $\quad p=1$ no, $p=2$ yes

$\lim_{R\to\infty}\int_1^R \frac{1}{x^p}\,dx = \lim_{R\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^R = \lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}$

$\lim R^{-p+1} = 0$ if $p>1$

$\qquad\qquad = \infty$ if $p<1$

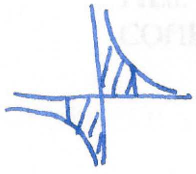
Integrals with discontinuities



$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is an improper integral! as $f(0)$ not defined.

$= \lim_{R\to 0}\int_R^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0}\left[2x^{1/2}\right]_R^1 = \lim_{R\to\infty} 2-2\sqrt{R} = 2$

Warning: $\int_{-1}^1 \frac{1}{x}\,dx \neq \left[\ln|x|\right]_{-1}^1 = \ln|1|-\ln|-1| = 0$ wrong!

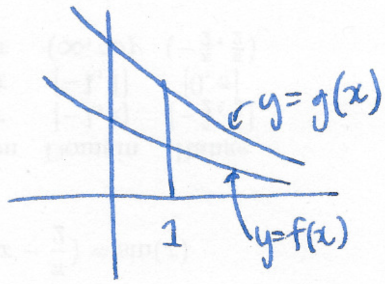


$[-1,1]$ contains discontinuity for $\frac{1}{x}$ so $\int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$

$$= \lim_{R\to 0} \int_{-1}^{R} \frac{1}{x}\,dx + \lim_{R\to 0} \int_{R}^{1} \frac{1}{x}\,dx = \lim_{R\to 0} \Big[\ln|x|\Big]_{-1}^{R} + \lim_{R\to 0} \Big[\ln|x|\Big]_{R}^{1}$$

$$= \lim_{R\to 0} \ln|R| - 0 \qquad \text{DNE.}$$

<u>Comparison test</u>



suppose $0 \le f(x) \le g(x)$ on $[1,\infty)$

if $\int_1^\infty g(x)\,dx$ converges

then $\int_1^\infty f(x)\,dx$ converges

<u>Example</u> $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ note $\sqrt{x^3+1} \ge \sqrt{x^3}$ on $[1,\infty)$

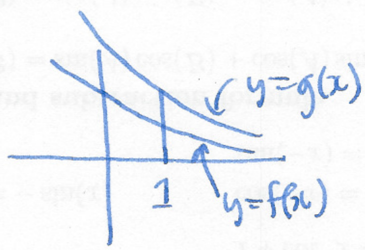
so $0 \le \frac{1}{\sqrt{x^3+1}} \le \frac{1}{\sqrt{x^3}}$

try $\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \int_1^\infty x^{-3/2}\,dx = \lim_{R\to\infty} \Big[-2x^{-1/2}\Big]_1^R = \lim_{R\to\infty} \frac{-2}{\sqrt{R}} + 2 = 2 < \infty$

converges.

$\Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges. (but don't know exact value!)

<u>other way</u>



$0 \le f(x) \le g(x)$ on $[1,\infty)$

if $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow \int_1^\infty g(x)\,dx$ diverges.

<u>note</u>: $\int_1^\infty g(x)\,dx$ diverges $\not\Rightarrow \int_1^\infty f(x)\,dx$ diverges

$\int_1^\infty f(x)\,dx$ converges $\not\Rightarrow \int_1^\infty g(x)\,dx$ converges.

<u>Example</u> show $\int_2^\infty \frac{1}{x-\sqrt{x}}\,dx$ diverges.

$x - \sqrt{x} < x$

$\frac{1}{x-\sqrt{x}} > \frac{1}{x}$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \Big[\ln|x|\Big]_1^R = \lim_{R\to\infty} \ln|R| - 0 \to \infty.$$