a even, b odd : write as powers of $\sec(x)$ and use integration by parts.

<u>Example</u> $\int \sin(3x)\cos(2x)\,dx$ useful fact $\sin(A+B) = \sin A\cos B + \cos A\sin B$ ①

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ ②

①+②: $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$$\int \sin(3x)\cos(2x)\,dx = \frac{1}{2}\int \sin(5x) + \sin(x)\,dx = -\frac{1}{10}\cos(5x) - \frac{1}{2}\cos(x) + c.$$

<u>Example</u> $\int \cos(4x)\cos(7x)\,dx$ $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

so $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$$= \frac{1}{2}\int \cos(11x) + \cos(-3x)\,dx = \frac{1}{22}\sin(11x) + \frac{1}{6}\cos(3x) + c$$

<u>§7.3 Trig substitutions</u>

aim: deal with $\sqrt{a^2-x^2}$ ①, $\sqrt{a^2+x^2}$ ②, $\sqrt{x^2-a^2}$ ③

$\cos^2 x + \sin^2 x = 1 \leftrightarrow \sin^2 x = 1 - \cos^2 x$ ①

$\leftrightarrow 1 + \tan^2 x = \sec^2 x$ ②

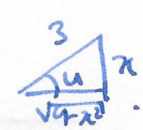
$\leftrightarrow \cot^2 x + 1 = \operatorname{cosec}^2 x \leftrightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$ ③

<u>Example</u> $\int \sqrt{9-x^2}\,dx$ try $x = 3\sin u$, $\frac{dx}{du} = 3\cos u$

$\cos 2u = \cos^2 u - \sin^2 u$
$= 2\cos^2 u - 1.$

$$\int \sqrt{9-9\sin^2 u}\,\frac{dx}{du}\,du = \int 3\sqrt{1-\sin^2 u}\;3\cos u\,du = \int 9\sqrt{\cos^2 u}\cdot\cos u\,du$$

$$= 9\int \cos^2 u\,du = \frac{9}{2}\int \cos 2u + 1\,du = \frac{9}{4}\sin 2u + \frac{9}{2}u + c.$$



$$= \frac{9}{4}\cdot 2\sin u\cos u + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) = \frac{9}{2}x\sqrt{1-\left(\frac{x}{3}\right)^2} + \frac{9}{2}\sin^{-1}(u) + c$$

$\frac{x}{3} = \sin u$

<u>Example</u> $\int \sqrt{1+4x^2}\, dx$ try: $x = \frac{1}{2}\tan u$

$$\frac{dx}{du} = \frac{1}{2}\sec^2 u$$

$$\int \sqrt{1+4\left(\tfrac{1}{2}\tan u\right)^2} \cdot \frac{dx}{du}\, du = \int \sqrt{1+\tan^2 u}\, \frac{1}{2}\sec^2 u\, du = \frac{1}{2}\int \sec^3 u\, du$$

$$= \frac{1}{2}\int \underbrace{\sec u}_{u} \cdot \underbrace{\sec^2 u}_{v'}\, du \qquad u = \sec u \quad u' = \sec u \tan u$$

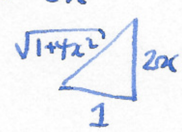
$$v' = \sec^2 u \quad v = \tan u$$

$$= \frac{1}{2}\sec u \tan u - \frac{1}{2}\int \sec u \tan^2 u\, du$$

$$\tan^2 u = \sec^2 u - 1$$

$$\frac{1}{2}\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\int \sec u\, du - \frac{1}{2}\int \sec^3 u\, du$$

$\tan u = 2x$



$$\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C.$$

$$\frac{1}{2}\int \sec^3 u\, du = \frac{1}{4}\sec u \tan u + \frac{1}{4}\ln|\sec x + \tan x| + C.$$

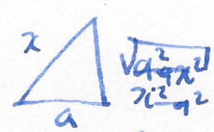
$$\frac{1}{4}\cdot\frac{1}{\sqrt{1+4x^2}}\, 2x + \frac{1}{4}\ln\left|\frac{1}{\sqrt{1+4x^2}} + 2x\right| + c$$

<u>Example</u> $\int \frac{1}{x^2\sqrt{x^2-a^2}}\, dx$ try: $x = a\sec u$

$$\frac{dx}{du} = a\sec u \tan u$$

$$\int \frac{1}{a^2\sec^2 u}\, \frac{1}{\sqrt{a^2\sec^2 u - a^2}}\, \frac{dx}{du}\, du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\cdot\frac{1}{\tan u}\, a\sec u \tan u\, du$$

$$= \frac{1}{a^2}\int \cos u\, du = \frac{1}{a^2}\sin u + C = \frac{1}{a^2}\,\frac{\sqrt{a^2-x^2}}{a} + C$$



$\sec u = \frac{x}{a}.$

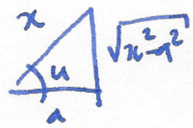
$$= \frac{1}{a^2}\sqrt{1-(x/a)^2} + c.$$

Example $\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$ try $x = a\sec u$

$$\frac{dx}{du} = a\sec u\tan u$$

$$\int \frac{1}{a^2\sec^2 u\sqrt{a^2\sec^2 u-a^2}}\frac{dx}{du}\,du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\,\frac{1}{\tan u}\,a\sec u\tan u\,du$$

$$= \frac{1}{a^2}\int \cos u\,du = \frac{1}{a^2}\sin u + c$$



$\frac{x}{a} = \sec u$ $\cos u = \frac{a}{x}$

$$= \frac{1}{a^2}\frac{\sqrt{x^2-a^2}}{x} = \frac{1}{a^2}\sqrt{1-(a/x)^2} + c.$$

§7.6 Partial fractions

aim: evaluate $\int \frac{P(x)}{Q(x)}\,dx$ $P(x), Q(x)$ polynomials.

recall: $\int \frac{1}{x+a}\,dx = \ln|x+a| + c$

special case: $\deg(P) < \deg(Q)$ and $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i all distinct real numbers.

then $\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \cdots + \frac{A_n}{x+a_n}$ A_i numbers.

Example $\int \frac{x}{x^2-1}\,dx$ $x^2-1 = (x-1)(x+1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x-1)(x+1)} = \frac{x(A+B) + B-A}{x^2-1}.$$

$$\left.\begin{aligned} A+B &= 1 \quad ① \\ -A+B &= 0 \quad ② \end{aligned}\right\}\quad ①+② \quad 2B = 1 \quad B = \tfrac{1}{2},\ A = \tfrac{1}{2}.$$

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$$

special case: as above, but $\deg(P) \geq \deg(Q)$ long division **Example** $\int \frac{x^3}{x^2-1}\,dx.$