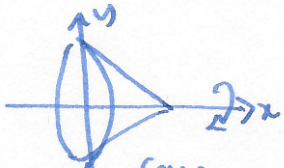
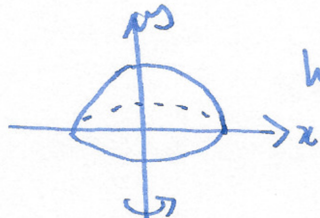
§6.3 Volumes of revolution

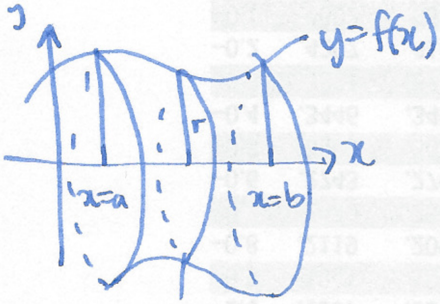
Examples



cone



half sphere



$A = \pi r^2$

recall: $V = \int_a^b A(x)\,dx$

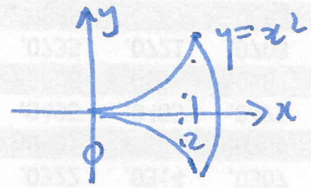
$A(x)$ = area of vertical cross-section

$= \pi r^2 = \pi y^2 = \pi (f(x))^2$

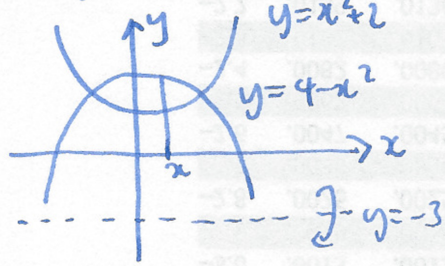
so volume $V = \int_a^b \pi (f(x))^2\,dx$

Example rotate $y = x^2$ about x axis.

$\pi \int_0^2 (x^2)^2\,dx = \pi \left(\frac{1}{5}x^5\right)_0^2 = \frac{32\pi}{5}$.



Example rotate region between $f(x) = x^2+2$ about $y = -3$
$f(x) = 4 - x^2$



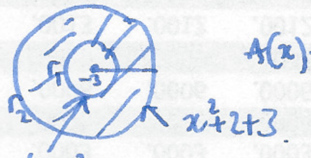
$V = \int_a^b A(x)\,dx$
$\quad\quad\quad\;\; \text{"}\pi r^2\text{"}$

a, b: find intersections

$x^2 + 2 = 4 - x^2$

$2x^2 = 2 \qquad x = \pm 1$

$A(x)$ = area of cross section



$A(x) = \pi r_2^2 - \pi r_1^2 = \pi (r_2^2 - r_1^2)$

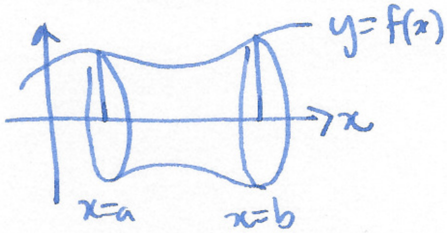
$A(x) = \pi \left[(4 - x^2 + 3)^2 - (x^2+5)^2 \right] = \pi \int_{-1}^{1} 24 - 4x^2\,dx$

$(7-x^2)^2 \qquad (x^2+5)^2$

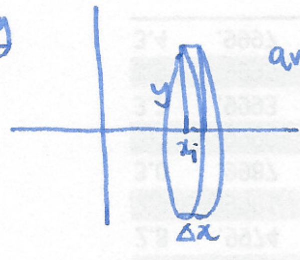
$49 - 14x^2 + \cancel{x^4} - (\cancel{x^4} + 10x^2 - 25) = \pi \left[24x - \frac{4}{3}x^3 \right]_{-1}^{+1} = \left(48 - \frac{8}{3}\right)\pi$.

$24 - 4x^2$

Surface area:



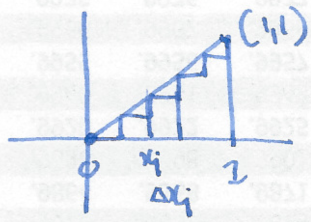
wrong



area of cylinder $= 2\pi y_i \Delta x_i$

surface area $= \sum 2\pi y_i \Delta x_i = 2\pi \int_a^b y\,dx$

analogy: arc length:

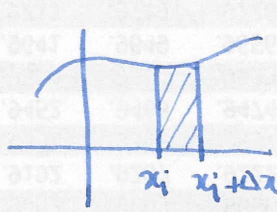


arc length $\neq \sum \Delta x_i$

$\neq \sum \Delta x_i + \Delta y_i$

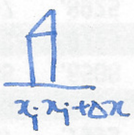
$\approx \sum \sqrt{\Delta x_i^2 + \Delta y_i^2}$.

what goes wrong:



area of strip $\approx y(x_i)\Delta x_i$ $\frac{\text{error}}{\text{area}} \to 0$ as $\Delta x_i \to 0$

surface area is like arc length:

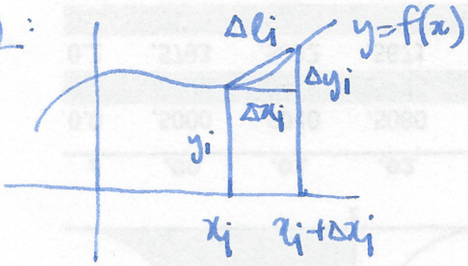


$\frac{\text{error}}{\text{arc length}} \not\to 0$ as $\Delta x_i \to 0$

↑ in fact constant

similarly $\frac{\text{error}}{\text{surface area}} \not\to 0$ as $\Delta x_i \to 0$.

right way:



arc length $\approx \sum \Delta l_i$

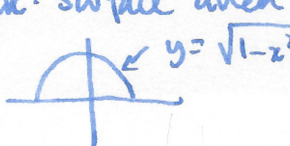
$(\Delta l_i)^2 = (\Delta x_i)^2 + (\Delta y_i)^2$

surface area of strip $\approx 2\pi y_i \Delta l_i$

total surface area: $\sum 2\pi y_i \Delta l_i = \sum 2\pi y_i(x_i)\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ ← doesn't look like an integral

$= 2\pi \sum y_i(x_i) \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\,\Delta x$ → surface area $S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$.

check: surface area of sphere



$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$

surface area $= 2\pi \int_{-1}^{1} \sqrt{1-x^2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$.

$= 4\pi$.