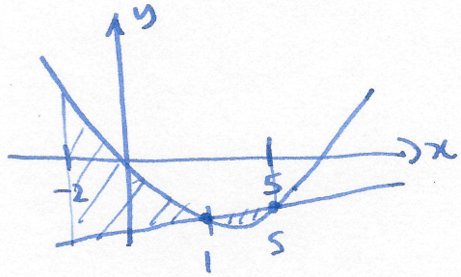
Examples

- find area between graphs $f(x) = x^2 - 5x - 7$ on $[-2, 5]$ and $g(x) = x - 12$

draw picture:



← need to break into two integrals

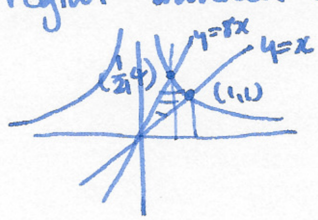
intersection points: $x^2 - 5x - 7 = x - 12$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1)$

- find area of region bounded by $y = \frac{1}{x^2}$, $y = x$, $y = 8x$

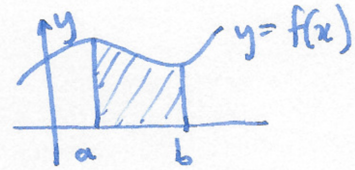
draw picture:



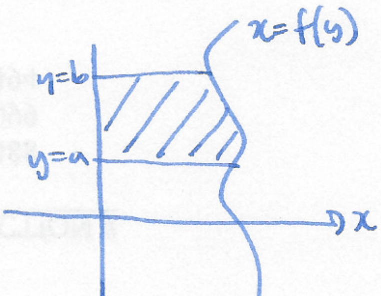
← need to break into 2 integrals.

Integration along y-axis

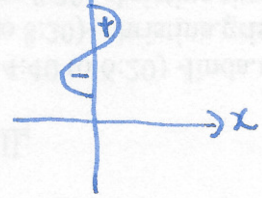
recall: $\int_a^b f(x)\,dx$



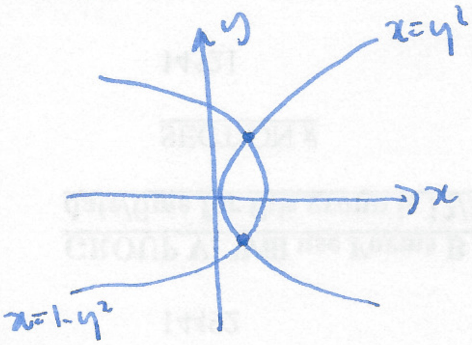
$\int_a^b f(y)\,dy$



signed area:



Example find area between the curves $x = y^2$ and $x = 1 - y^2$



intersection points $y^2 = 1 - y^2$ $\quad 2y^2 = 1$ $\quad y = \pm\frac{1}{\sqrt{2}}$

$$\int_{-1/\sqrt{2}}^{+1/\sqrt{2}} 1 - y^2 - y^2 \, dy$$

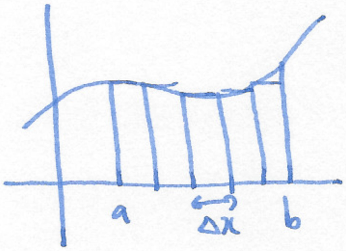
§6.2 Volume, average value

recall: volume of cylinder is Ah  A = area of base

Fact: this works for cylinders of any base shape:  $V = Ah$

suppose the shape is not a cylinder:  can approximate by horizontal slices.

recall



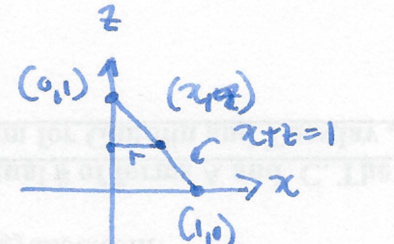
$\leftarrow$ can approximate area under curve by rectangles of width Δx

$\leftarrow$ can approximate volume of object by cylinders of ~~width~~ height Δy

volume $V = \sum V_i \approx \sum A(y_i)\Delta y$

volume $V = \int_a^b A(y)\,dy$

Example Find volume of cone: $A = \pi r^2$



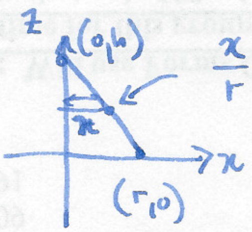
$r = x$

$$V = \int_0^1 A(z)\,dz = \int_0^1 \pi r^2\,dz = \int_0^1 \pi x^2\,dz = \int_0^1 \pi(1-z)^2\,dz$$

$1-2z+z^2$

$$= \pi\left[z - z^2 + \tfrac{1}{3}z^3\right]_0^1 = \pi\left(1 - 1 + \tfrac{1}{3}\right) = \frac{\pi}{3}.$$

in general:



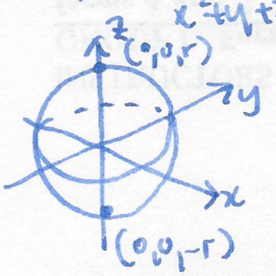
$\frac{x}{r} + \frac{z}{h} = 1 \qquad x = r\left(1 - \frac{z}{h}\right)$

$$V = \int_0^h A(z)\,dz = \int_0^h \pi x^2\,dz = \int_0^h \pi r^2\left(1-\frac{z}{h}\right)^2 dz = \pi r^2\int_0^h 1 - \frac{2z}{h} + \frac{z^2}{h^2}\,dz$$

$$= \pi r^2\left[z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right]_0^h = \pi r^2\left(h - h + \frac{h}{3}\right) = \frac{1}{3}\pi r^2 h.$$

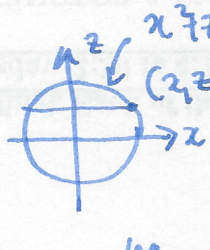
Example Volume of sphere

$x^2+y^2+z^2=r^2$



$$V = \int_{-r}^{r} A(z)\,dz$$

$x^2+z^2=r^2$



$x = \sqrt{r^2 - z^2}$

$$V = \int_{-r}^{r} \pi(r^2 - z^2)\,dz = \pi\left[r^2 z - \frac{1}{3}z^3\right]_{-r}^{r} = 2\pi\left(r^3 - \frac{1}{3}r^3\right)$$

$$= \frac{4}{3}\pi r^3.$$