Math 232 Calculus 2 Spring 18 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find $\int \sin 4x \sin 5x \, dx$.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$

$= \frac{1}{2} \int \cos x - \cos 9x \, dx = \frac{1}{2} \sin x - \frac{1}{18} \sin 9x + c$

(2) (10 points) Find $\int \sec^3 x \, dx.$ $= \int \sec x \cdot \sec^2 x \, dx$

$\int uv' dx = uv - \int u'v \, dx$ $\quad u = \sec x \quad u' = \sec x \tan x \quad \sin^2 x + \cos^2 x = 1$

$v' = \sec^2 x \quad v = \tan x \quad \tan^2 x + 1 = \sec^2 x$

$= \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx$

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + c$

$\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$

$\sin^2 x + \cos^2 x = 1$

(3) (10 points) Find $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

$x = 2\sin u$

$\cancel{\frac{du}{dx} = 2\cos x}$ $\quad \frac{dx}{du} = 2\cos u$

$$\int \frac{4\sin^2 u}{\sqrt{4-4\sin^2 u}} \cdot 2\cos u\,du = \int \frac{4\sin^2 u \cdot \cos u}{\cos u}\,du = \int 4\sin^2 u\,du$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$$= \int 2 - 2\cos 2u\,du = 2u - \sin 2u + c$$

$$= \sin^{-1}\left(\frac{x}{2}\right) - 2\sin u\cos u + c = \sin^{-1}\left(\frac{x}{2}\right) - \frac{x\cdot\sqrt{4-x^2}}{2} + c$$

(4) (10 points) Find $\displaystyle\int \frac{x^2 - x}{(x+1)(x^2+1)}\,dx.$

$$\frac{x^2-x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$x=-1$: $\quad 2 = 2A \Rightarrow A = 1$

$x = 0$: $\quad 0 = A + C \Rightarrow C = -1$

$x = 1$: $\quad 0 = 2A + 2B + 2C \Rightarrow B = 0$

$$= \int \frac{1}{x+1}\,dx + \int \frac{-1}{x^2+1}\,dx = \ln|x+1| - \tan^{-1}(x) + C$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(5) (10 points) Find $\int_0^\infty xe^{-3x}\,dx$.

$$\int xe^{-3x}\,dx \qquad u = x \quad u' = 1 \qquad v' = e^{-3x} \quad v = -\frac{1}{3}e^{-3x}$$

$$= -\frac{x}{3}e^{-3x} - \int -\frac{1}{3}e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c$$

$$\lim_{R\to\infty}\int_0^R xe^{-3x}\,dx = \lim_{R\to\infty}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]_0^R$$

$$= \lim_{R\to\infty} -\frac{1}{3}Re^{-3R} - \frac{1}{9}e^{-3R} + 0 + \frac{1}{9} = \frac{1}{9}$$

(6) (10 points) Find the degree three Taylor polynomial for $f(x) = e^{1/x}$ centered at $x = 1$.

$f(x) = e^{x^{-1}}$ $\quad f(1) = e$

$f'(x) = e^{x^{-1}} \cdot -x^{-2}$ $\quad f'(1) = -e$

$f''(x) = e^{x^{-1}} \cdot x^{-4} + e^{x^{-1}} \cdot 2x^{-3}$ $\quad f''(1) = 3e$

$f^{(3)}(x) = e^{x^{-1}} \cdot -x^{-6} + e^{x^{-1}} \cdot -4x^{-5} + e^{x^{-1}} \cdot -2x^{-5} + e^{x^{-1}} \cdot -6x^{-4}$ $\quad f^{(3)}(1) = -13e$

$T_3(x) = e - e(x-1) + 3e\frac{(x-1)^2}{2!} + -13e\frac{(x-1)^3}{3!}$

(7) Consider the series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} \cdots$. Find an explicit formula for the partial sum S_N. Hint: compare S_N with $\frac{2}{3}S_N$. Use your formula for S_N to show that the series converges, and find the limit.

$$S_N = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{N-1}$$

$$\frac{2}{3}S_N = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{N-1} + \left(\frac{2}{3}\right)^N$$

$$S_N - \frac{2}{3}S_N = 1 - \left(\frac{2}{3}\right)^N$$

$$\frac{1}{3}S_N = 1 - \left(\frac{2}{3}\right)^N \qquad S_N = 3\left(1 - \left(\frac{2}{3}\right)^N\right)$$

$$\lim_{N\to\infty} 3\left(1 - \left(\frac{2}{3}\right)^N\right) = 3.$$

(8) Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ converge or diverge? If it converges, find the exact value.

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$$

$n=-1$: $1 = A$

$n=-2$: $1 = -B$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_N = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{N+1} - \frac{1}{N+2} = \frac{1}{2} - \frac{1}{N+2}$$

$$\lim_{N\to\infty} \frac{1}{2} - \frac{1}{N+2} = \frac{1}{2}$$

(9) Does the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ converge or diverge? Explain.

$$1 + \underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}} + \cdots$$

diverges.

(10) Does the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ converge or diverge?

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ so diverges.