Calc 2 Sample midterm solutions

Q1 $\int \tan^3 2x\,dx \qquad u = 2x \quad \frac{du}{dx} = 2 \qquad \int \tan^3 u \,\frac{1}{2}\,du = \frac{1}{2}\int \tan u\,(1-\sec^2 u)\,du$

$v = \tan u \quad \frac{dv}{du} = \sec^2 u \qquad \frac{1}{2}\int \tan u\,du - \frac{1}{2}\int v.\sec^2 u.\frac{1}{\sec^2 u}.\,dv$

$= \frac{1}{2}\ln|\sec u| - \frac{1}{2}\int v\,dv = \frac{1}{2}\ln|\sec u| - \frac{1}{4}v^2 + c$

$= \frac{1}{2}\ln|\sec(2x)| - \frac{1}{4}\tan^2(2x) + c$

Q2 $\int \cos 11x\,.\,\sin 7x\,dx \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

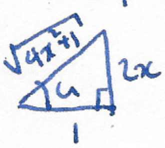
$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$\frac{1}{2}\int \sin 18x + \sin(-4x)\,dx = -\frac{1}{2}\cdot\frac{1}{18}\cos 18x + \frac{1}{2}\cdot\frac{1}{4}\cos 4x + C$

Q3 $\int \frac{x}{\sqrt{4x^2+1}}\,dx \qquad \sin^2 x + \cos^2 x = 1 \qquad x = \frac{1}{2}\tan u$

$\tan^2 x + 1 = \sec^2 x \qquad \frac{dx}{du} = \frac{1}{2}\sec^2 u$

$\int \frac{\frac{1}{2}\tan u}{\sqrt{4.\frac{1}{4}\tan^2 u + 1}}\cdot\frac{1}{2}\sec^2 u\,du = \frac{1}{4}\int \frac{\tan u.\sec^2 u}{\sec u}\,du = \frac{1}{4}\int \tan u.\sec u\,du = \frac{1}{4}\sec u + c$



$= \frac{\sqrt{4x^2+1}}{2x} + c$

Q4 $\int \frac{5x+4}{(x-2)(x+2)^2}\,dx \qquad \frac{5x+4}{(x-2)(x+2)^2} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$= \frac{A(x+2)^2 + B(x-2)(x+2) + C(x-2)}{(x-2)(x+2)^2}$

$x=2: \; 14 = 16A \qquad A = 7/8$

$x=-2: \; -6 = -4C \qquad C = 3/2$

$x=0: \; 4 = 4A - 4B - 2C$

$4 = \frac{7}{2} - 4B - 3 \qquad 4B = \frac{7}{2} - 7 = -\frac{7}{2} \qquad B = -\frac{7}{8}$

$$\int \frac{7/8}{x-2} + \frac{-7/8}{x+2} + \frac{3/2}{(x+2)^2}\,dx = \frac{7}{8}\log|x-2| - \frac{7}{8}\log|x+2| - \frac{3}{2}\cdot\frac{1}{x+2} + C$$

Q5 $2\int \underbrace{x^2}_{v}\,\underbrace{\ln x}_{u}\,dx$ $\qquad \int uv'\,dx = uv - \int u'v\,dx$ $\qquad u = \ln x \quad u' = \frac{1}{x}$, $\quad v' = x^2 \quad v = \frac{1}{3}x^3$

$$= 2\left(\frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3\cdot\frac{1}{x}\,dx\right) = 2\left(\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c\right).$$

$$2\int_0^1 x^2\ln x\,dx = \lim_{R\to 0} 2\int_R^1 x^2\ln x\,dx = \lim_{R\to 0} 2\left[\frac{1}{3}x^3\ln x - \frac{1}{9}x^3\right]_R^1$$

$$= 2\lim_{R\to 0} -\frac{1}{9} - \frac{1}{3}R^3\ln R + \frac{1}{9}R^3 \qquad \lim_{R\to 0} R^3\ln R = \lim_{R\to 0}\frac{\ln R}{R^{-3}} \underset{\text{l'H.}}{=} \lim_{R\to 0}\frac{1/R}{-3R^{-4}}$$

$$= -\frac{2}{9} \qquad\qquad = \lim_{R\to 0} -\frac{1}{3}R^3 = 0.$$

Q6 $\int_0^\infty \frac{1}{16+x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{16+x^2}\,dx$ $\qquad x = 4u$, $\frac{dx}{du} = 4$ $\qquad \int \frac{1}{16+16u^2}\cdot 4\,du = \frac{1}{4}\int\frac{1}{1+u^2}\,du = \frac{1}{4}\tan^{-1}(u)$

$$= \lim_{R\to\infty}\left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right]_0^R = \lim_{R\to\infty}\frac{1}{4}\tan^{-1}\left(\frac{R}{4}\right) - 0 = \frac{\pi}{8}.$$

Q7 No: $f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ not defined at $x=0$.

$f(x) = x^{1/2}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(1) = 1/2$ $\qquad T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{8}\frac{(x-1)^2}{3!}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f''(1) = -1/4$

$f^{(3)}(x) = +\frac{3}{8}x^{-5/2}$ $\qquad f^{(3)}(1) = 3/8$

see last page for error bound

Q8 $a_n = \frac{3^n}{n!} = \frac{3.3.3.}{1.2.3}\underbrace{\frac{3}{4}\frac{3}{5}\cdots\frac{3}{n-1}}_{<1}\cdot\frac{3}{n} < \frac{9}{2}\cdot\frac{3}{n}$ $\quad$ so $\quad 0 < a_n < \frac{27}{2}\frac{1}{n}$

squeeze thm $\Rightarrow$ $a_n \to 0$ as $\frac{27}{2}\cdot\frac{1}{n}\to 0$.

Q9 geometric series: $c + cr + cr^2 + \dots = \frac{c}{1-r}$ for $|r|<1$ $\qquad \sum_{n=2}^{\infty} e^{-n} = \frac{e^{-2}}{1-e^{-1}} = \frac{1}{e^2-e}$

Q10 $\frac{1}{n^2+3n+2} = \frac{1}{(n+2)(n+1)} = \frac{A}{n+2} + \frac{B}{n+1} = \frac{A(n+1)+B(n+2)}{(n+2)(n+3)}$

$n=-1: \quad 1 = B$ $\qquad \dfrac{1}{n^2+3n+2} = \dfrac{1}{n+1} - \dfrac{1}{n+2}$

$n=-2: \quad 1 = -A$

so $\quad S_N = \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{3} - \dfrac{1}{4} + \dots + \dfrac{1}{N+1} - \dfrac{1}{N+2} = \dfrac{1}{2} - \dfrac{1}{N+2}$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2} - \frac{1}{N+2} = \frac{1}{2}.$$

__Q11__ $\quad \cos\left(\frac{1}{n}\right) \to 1$ as $n \to \infty$ so $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges.

__Q12__ $\quad f(x) = \cancel{x^{1/2}}\ x^2-2$ $\qquad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^2-2}{2x_n}$

$f'(x) = \cancel{\tfrac{1}{2}x^{-1/2}}\ 2x$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + 2}{2x_n} = \frac{x_n^2+2}{2x_n}$$

if $\quad x_n > \sqrt{2}$ then $\quad x_n^2 > 2$

$2x_n^2 > 2 + x_n^2$

$x_n > \dfrac{2+x_n^2}{2x_n} = x_{n+1}$. $\quad$ so $\quad x_n$ decreasing

$\Rightarrow x_{n+1} \geq 0$

x_n ~~always~~ positive ~~& so~~ so $x_n \geq 0$ for all n. If $0 < x_n < \sqrt{2}$, argument above shows x_n increasing, so in fact $x_n \geq \sqrt{2}$.

~~As~~ decreasing, bounded below, so limit exists, $\displaystyle\lim_{n\to\infty} x_n = l. = \lim_{n\to\infty} x_{n+1}$

$$\lim_{n\to\infty} \frac{2+x_n^2}{x_n} = \frac{2 + \lim_{n\to\infty} x_n^2}{\lim_{n\to\infty} x_n} = \frac{2+l^2}{l} = l \Rightarrow \quad 2+l^2 = 2l^2$$

$2 = l^2.$

$l = \sqrt{2}.$

__Q7__ $\quad |T_3(2) - \sqrt{2}| \leq K\dfrac{|2-1|^4}{4!}$ where K is an upper bound for $|f^{(4)}(x)|$ on $[1,2]$

$f^{(4)}(x) = -\dfrac{15}{16}x^{-7/2}$ $\qquad$ so $\quad |f^{(4)}(x)| \leq +\dfrac{15}{16}$ on $[1,2]$.

so $\quad |T_3(2) - \sqrt{2}| \leq \dfrac{15}{16 \cdot 4!}$