Calc2 SMT1 Solutions

Q1 $\displaystyle\int \frac{\sin x}{1+\cos x}\,dx$ $\quad u = \cos x + 1$, $\quad \frac{du}{dx} = -\sin x$

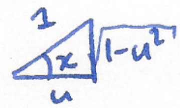
$$\int \frac{\sin x}{u}\cdot\frac{dx}{du}\,du = \int \frac{\sin x}{u}\cdot\frac{1}{-\sin x}\,du$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln|1+\cos x| + c$$

Q2 $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$ $\quad u = \cos x$, $\quad \frac{du}{dx} = -\sin x$

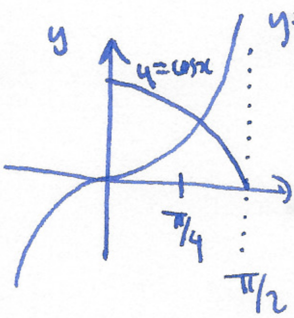
$$\int \frac{\sin x}{1+u^2}\cdot\frac{dx}{du}\,du = \int \frac{\sin x}{1+u^2}\cdot\frac{1}{-\sin x}\,du$$

$$= -\int \frac{1}{1+u^2}\,du = -\tan^{-1}(u) + c$$



$$= -\tan^{-1}(\cos x) + c$$

Q3



find intersection point: $\quad 2\tan x = \cos x$

$$\frac{2\sin x}{\cos x} = \cos x$$

$$2\sin x = \cos^2 x = 1 - \sin^2 x$$

$$\sin^2 x + 2\sin x - 1 = 0$$

total area: $\displaystyle\int_0^a \cos x - 2\tan x\,dx + \int_a^{\pi/4} 2\tan x - \cos x\,dx$

$$\sin x = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

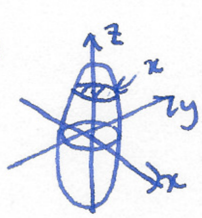
$$x \simeq 0.43 < \frac{\pi}{4} \approx 0.79$$

where $a = \sin^{-1}(\sqrt{2}-1)$

$$\int_0^a \cos x - 2\tan x\,dx = \Big[\sin x - 2\ln|\sec x|\Big]_0^a + \Big[2\ln|\sec x| - \sin x\Big]_a^{\pi/4}.$$

$$= \sin(a) - 2\ln|\sec(a)| + 2\ln\left|\frac{2}{\sqrt{2}}\right| - \frac{\sqrt{2}}{2} - 2\ln|\sec(a)| + \sin(a) \approx \cancel{+16}\ 1.15.$$

Q4 a)



$4y^2 + 4z^2 = 1 - x^2$ ← circle of radius $\sqrt{\frac{1-x^2}{4}}$

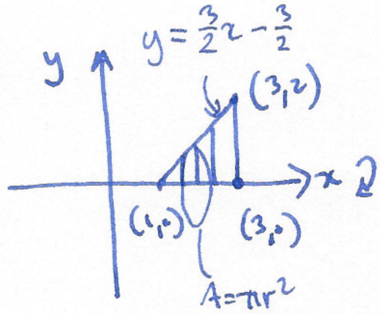
area $\pi r^2 = \pi(1-x^2)/4$

b) $\displaystyle V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \frac{\pi}{4}(1-x^2)\,dx = \frac{\pi}{4}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{\pi}{4}\left(1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right)$

$$= 3\pi$$

Q5 $\frac{1}{5}\int_0^5 e^{-x/5}\,dx = \frac{1}{5}\left[-\frac{1}{5}e^{-x/5}\right]_0^5 = -\frac{1}{25}\left(e^{-1}-e^{0}\right) = \frac{1-1/e}{25}.$

Q6

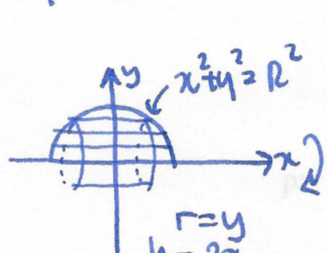


$$\int_1^3 \pi y^2\,dx = \frac{3^2\pi}{2^2}\int_1^3 (x-1)^2\,dx = \frac{9\pi}{4}\int_1^3 x^2-2x+1\,dx$$

$$= \frac{9\pi}{4}\left[\frac{1}{3}x^3-x^2+x\right]_1^3 = \frac{9}{4}\pi\left(9-9+3-\left(\frac{1}{3}-1+1\right)\right)$$

$$= \frac{9}{4}\pi\frac{8}{3} = 6\pi$$

Q7



$$V = \int_0^R 2\pi rh\,dy = \int_0^R 2\pi y\cdot 2\sqrt{R^2-y^2}\,dy$$

$r = y$
$h = 2x$

$$= 4\pi\int_0^R y\sqrt{R^2-y^2}\,dy \qquad u = R^2-y^2,\quad \frac{du}{dy} = -2y \qquad = 4\pi\int_{R^2}^0 y\cdot\sqrt{u}\,\frac{dy}{du}\,du$$

$$= 4\pi\int_{R^2}^0 y\sqrt{u}\cdot\frac{1}{-2y}\,du = 2\pi\int_0^{R^2}\sqrt{u}\,du = 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{R^2} = \frac{4}{3}\pi R^3.$$

Q8

$$\int \underbrace{x^2}_{v'}\underbrace{\ln(x+2)}_{u}\,dx \qquad u = \ln(x+2) \quad u' = \frac{1}{x+2} \qquad v' = x^2 \quad v = \frac{1}{3}x^3$$

$$= \frac{1}{3}x^3\ln(x+2) - \int \frac{1}{3}x^3\frac{1}{x+2}\,dx \quad (*)$$

$u = x+2$
$\frac{du}{dx} = 1$

$$\int \frac{x^3}{x+2}\,dx = \int \frac{(u-2)^3}{u}\,du = \int u^2-6u+6-\frac{8}{u}\,du = \frac{1}{3}u^3-3u^2+6u-8\ln|u|+c$$

so $(*) = \frac{1}{3}x^3\ln(x+2) + \frac{1}{3}(x+2)^3 - 3(x+2)^2 + 6(x+2) - 8\ln|x+2| + c$

Q9

$$\int \underbrace{e^{-2x}}_{u}\underbrace{\sin(3x)}_{v'}\,dx = -e^{-2x}\frac{1}{3}\cos(2x) - \int \underbrace{\frac{2}{3}e^{-2x}}_{u}\underbrace{\cos(3x)}_{v'}\,dx$$

$u = e^{-2x}$ $u' = -2e^{-2x}$
$v' = \sin(3x)$ $v = -\frac{1}{3}\cos(3x)$

$u = \frac{2}{3}e^{-2x}$ $u' = -\frac{4}{3}e^{-2x}$
$v' = +\frac{1}{3}\cos(3x)$ $v = +\frac{1}{9}\sin(3x)$

$$\int e^{-2x}\sin(3x)\,dx = -\frac{1}{3}e^{-2x}\cos(2x) - \frac{2}{27}e^{-2x}\sin(3x) + \int -\frac{4}{3}e^{-2x}\frac{1}{9}\sin(3x)\,dx$$

$$\frac{31}{27}\int e^{-2x}\sin(3x)\,dx = -\frac{1}{3}e^{-2x}\cos(2x) - \frac{2}{27}e^{-2x}\sin(3x)+c$$

$$\int e^{-2x}\sin(3x)\,dx = -\frac{9}{31}e^{-2x}\cos(2x) - \frac{2}{31}e^{-2x}\sin(3x)+c$$

Q10

$$\int \underbrace{x}_{u}\underbrace{e^{-x}\cos(x)}_{v'}\,dx \qquad u = x \quad u' = 1$$

$$v' = \underbrace{e^{-x}}_{w}\underbrace{\cos(x)}_{z'}$$

$w = e^{-x} \quad w' = -e^{-x}$

$z' = \cos(x) \quad z = \sin x.$

$$\int e^{-x}\cos(x)\,dx = e^{-x}\sin x + \int \underbrace{+e^{-x}}_{w}\cdot\underbrace{\sin x}_{z'}\,dx \qquad w = e^{-x} \quad w' = -e^{-x}$$

$z' = \sin x \quad z = -\cos x$

$$\int e^{-x}\cos(x)\,dx = e^{-x}\sin x + e^{-x}(-\cos x) - \int -e^{-x}\cdot -\cos x\,dx$$

$$2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x$$

$$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}\sin x - \frac{1}{2}e^{-x}\cos x. = v.$$

$$\int xe^{-x}\cos x\,dx = x\cdot\frac{1}{2}e^{-x}(\sin x - \cos x) - \int \frac{1}{2}e^{-x}\sin x - \frac{1}{2}\underbrace{e^{-x}\cos x}_{\text{already done this one.}}\,dx \quad (*)$$

$$\int \underbrace{e^{-x}}_{u}\underbrace{\sin x}_{v'}\,dx \qquad u = e^{-x} \quad u' = -e^{-x}$$

$v' = \sin x \quad v = -\cos x$

$$= -e^{-x}\cos x - \int \underbrace{+e^{-x}}_{u}\cdot\underbrace{+\cos x}_{v'}\,dx \qquad u = e^{-x} \quad u' = -e^{-x}$$

$v' = \cos x \quad v = \sin x$

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - e^{-x}\sin x - \int -e^{-x}\sin x\,dx$$

$$2\int e^{-x}\sin x\,dx = -e^{-x}(\sin x + \cos x)$$

$$(*)\ \int xe^{-x}\cos x\,dx = \frac{1}{2}xe^{-x}(\sin x - \cos x) - \frac{1}{2}\cdot\left(-\frac{1}{2}\right)e^{-x}(\sin x + \cos x) + \frac{1}{2}\frac{1}{2}e^{-x}(\sin x - \cos x) + c$$

$$= \frac{1}{2}xe^{-x}(\sin x - \cos x) + \frac{1}{2}e^{-x}\sin x + c$$

Q11

$$\int_0^{\pi/2}\sin^3(x)\cos^2(x)\,dx = \int_0^{\pi/2}\sin x(1-\cos^2 x)\cos^2 x\,dx \qquad u = \cos x \quad \frac{du}{dx} = -\sin x$$

$$\int_1^0 \sin x\cdot(1-u^2)u^2\frac{dx}{du}\,du = \int_0^1 u^2 - u^4\,du = \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Q12 $\int \sin(4x)\cos(9x)\,dx$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$$

$$= \frac{1}{2}\int \sin(13x)+\sin(-5x)\,dx = -\frac{1}{2}\frac{1}{13}\cos(13x) + \frac{1}{10}\cos(5x) + C.$$