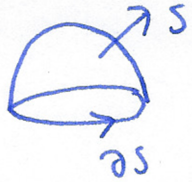
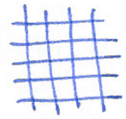
intuition

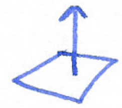


$$\oint_{\partial S} \underline{F} \cdot \underline{ds} = \iint_S \text{curl}(\underline{F}) \cdot \underline{dS}$$

divide S into small squares



same calculation as last time

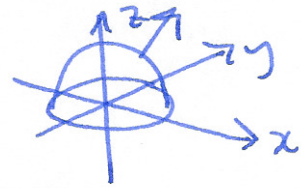


$\underline{n} = (0,0,1)$

$$(\underline{\nabla} \times \underline{F}) \cdot \underline{n} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \quad \square.$$

Example $\underline{F} = \langle -y, 2x, x+z \rangle$

S = upper hemisphere



$\oint_{\partial S} \underline{F} \cdot \underline{ds}$ $\quad c(\theta) = (\cos\theta, \sin\theta, 0) \quad 0 \leq \theta \leq 2\pi$

$c'(\theta) = (-\sin\theta, \cos\theta, 0)$

$$= \int_0^{2\pi} \langle -\sin\theta, 2\cos\theta, \cos\theta \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + 2\cos^2\theta \, d\theta = \int_0^{2\pi} 1 + \cos^2\theta \, d\theta = 2\pi + \pi = 3\pi.$$

$\iint_S \underline{\nabla} \times \underline{F} \, \underline{dS}$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix} = \langle 0, -1, 2+1 \rangle = \langle 0, -1, 3 \rangle$$

parameterization: $(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \quad \begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{matrix}$

$$\int_0^{\pi/2} \int_0^{2\pi} \langle 0, -1, 3 \rangle \cdot \underline{n} \, d\theta \, d\phi$$

$$\frac{\partial \Phi}{\partial \theta} = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0 \rangle$$

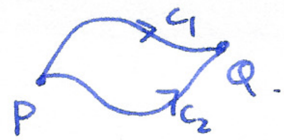
$$\frac{\partial \Phi}{\partial \phi} = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi \rangle$$

$$\underline{n} = \langle \cos\theta \sin^2\phi, \sin\theta \sin^2\phi, \sin^2\theta \sin\phi \cos\phi + \cos^2\theta \cos\phi \sin\phi \rangle$$

$$= \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$

$$\int_0^{\pi/2} \int_0^{2\pi} -\sin\theta \sin\phi + 3 \sin\phi \cos\phi \, d\phi = 3\pi.$$

recall : $F = \nabla f$ then $\int_{C_1} \underline{F}.\underline{ds} = \int_{C_2} \underline{F}.\underline{ds} = f(Q) - f(P)$



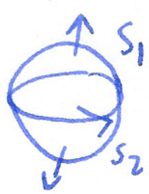
(path independence for gradient vector fields)

Stoke's Thm : $\oint_{\partial S} \underline{F}.\underline{ds} = \iint_S \nabla \times \underline{F}.\underline{dS}$ note: integral depends only on ∂S not S itself

i.e. surface independence for curl vector fields.

$\oint_{\partial S} \underline{F}.ds = \iint_{S_1} \nabla \times \underline{F}\ \underline{dS} = \iint_{S_2} \nabla \times \underline{F}.\underline{dS}$ if $\partial S_1 = \partial S_2$.

in particular, if S is closed $(\partial S = \phi)$ then $\iint_S \nabla \times \underline{F}.\underline{dS} = 0$

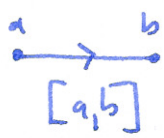


$S = S_1 \cup S_2$ with $\partial S_1 = -\partial S_2$ (reversed orientation).

§17.3 Divergence Thm

recall : we have theorems of the form: integral of a derivative on a domain = integral over oriented boundary of the domain

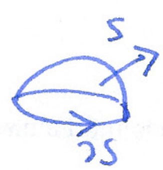
FTC : $\int_a^b f'(x)dx = f(b) - f(a)$



$\partial[a,b] = +b - a$

line integrals : $\int_c \underline{\nabla f}.\underline{ds} = f(Q) - f(P)$

Stoke's Thm : $\iint_S \nabla \times \underline{F}.\underline{dS} = \int_{\partial S} \underline{F}.ds$



Divergence Thm : $\iiint_W \operatorname{div}(\underline{F}).dV = \iint_{\partial W} \underline{F}.\underline{dS}$

outward pointing normal

Defn the divergence of a vector field $\underline{F}$ is $\underset{\underline{\nabla}.\underline{F}}{\underset{|||}{\operatorname{div}(\underline{F})}} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

scalar!

useful properties: $\nabla \cdot (\underline{F}+\underline{G}) = \nabla \cdot \underline{F} + \nabla \cdot \underline{G}$

$\nabla \cdot (c\underline{F}) = c \nabla \cdot \underline{F}$ (c constant)

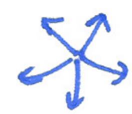
Example $\underline{F} = \langle x^2+y^2, xyz, e^{xyz} \rangle$

$\underline{\nabla} \cdot \underline{F} = 2x + xz + xye^{xyz}$

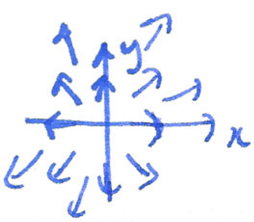
Intuition:  small ball B, so $\nabla \cdot \underline{F}$ approx constant $\partial B = S$

then flux across S $= \iiint_B \underline{\nabla} \cdot \underline{F} \, dV \approx \mathrm{div}(\underline{F}) \, \mathrm{vol}(B)$

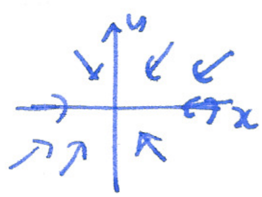
flux $\approx$ divergence x vol (locally)

- $\mathrm{div}(\underline{F}) > 0$ at a point: fluid flows out "source" 
- $\mathrm{div}(\underline{F}) < 0$: fluid flows in "sink" 
- $\mathrm{div}(\underline{F}) = 0$: no net flow in or out "incompressible flow"

Examples (2d)

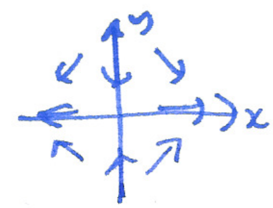


$\underline{F} = \langle x, y \rangle$ $\nabla \cdot \underline{F} = 2$



$\underline{F} = \langle -x, -y \rangle$

$\underline{\nabla} \cdot \underline{F} = -2$



$\underline{F} = \langle x, -y \rangle$

$\underline{\nabla} \cdot \underline{F} = 0$

Example (verify Stoke's)

$\underline{F} = \langle y, yz, z^2 \rangle$ S: $x^2+y^2 = 4$ $0 \le z \le 5$

$\underline{\nabla} \cdot \underline{F} = 0 + z + 2z = 3z$

$\iiint_W 3z \, dV = 150\pi$ (cylindricals).