warning: can only integrate vector fields over orientable surfaces

orientable  non-orientable:  Möbius band

Example find $\iint_S \underline{F}\cdot d\underline{S}$ $\quad \underline{F} = \langle 0,0,x\rangle$

$$S: \Phi(u,v) = (u^2, v, u^3 - v^2) \qquad 0 \le u \le 1, \quad 0 \le v \le 1$$

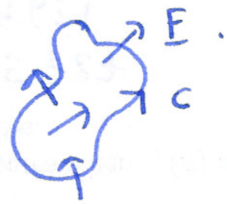
$$\frac{\partial \Phi}{\partial u} = \langle 2u, 0, 3u^2\rangle \qquad \frac{\partial \Phi}{\partial v} = \langle 0, 1, -2v\rangle$$

$$\underline{n} = \begin{vmatrix} i & j & k \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{vmatrix} = \langle -3u^2, 4uv, 2u\rangle$$

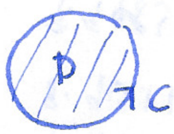
$$\iint_S \underline{F}\cdot d\underline{S} = \int_0^1\int_0^1 \underline{F}(\Phi(u,v))\cdot \underline{n}(u,v)\, dv\, du$$

$$= \int_0^1\int_0^1 \langle 0,0,u^2\rangle\cdot\langle -3u^2, 4uv, 2u\rangle\, dv\, du = \int_0^1\int_0^1 2u^3\, dv\, du = \left[\tfrac{1}{2}u^4\right]_0^1 = \tfrac{1}{2}$$

§17.1 Green's Theorem

recall: $\oint_C \nabla f\cdot d\underline{s} = 0$ 

suppose $\underline{F} \neq \nabla f$?

orientation convention: 

counter clockwise

Thm (Green's Thm) D domain in $\mathbb{R}^2$ s.t. $\partial D = C$ simple closed curve oriented clockwise, then $\oint_C \underline{F}\cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$ $\quad \underline{F} = \langle F_1, F_2\rangle$

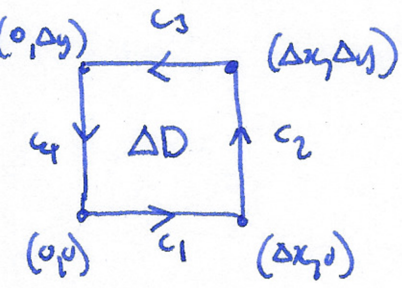
intuition 

note: 

$$\oint_{\partial D_1} + \oint_{\partial D_2} = \oint_{\partial(D_1 \cup D_2)}$$

so suffices to check on a very small square where $\underline{F}$ is almost linear.

$\underline{F} = \langle px+qy, rx+sy \rangle$



$c_1'(t) = \langle 1,0 \rangle$ etc.

$c_1: \int_0^{\Delta x} pt + q\cdot 0 \, dt = \left[\frac{1}{2}pt^2\right]_0^{\Delta x} = \frac{1}{2}p\Delta x^2$

$c_2: \int_0^{\Delta y} r\Delta x + st \, dt = \left[r\Delta x t + \frac{1}{2}st^2\right]_0^{\Delta y} = r\Delta x\Delta y + \frac{1}{2}s\Delta y^2.$

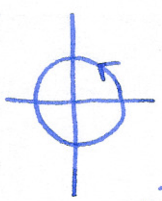
$c_3: -\int_0^{\Delta x} pt + q\Delta y \, dt = -\left[\frac{1}{2}pt^2 + q\Delta y t\right]_0^{\Delta x} = -\frac{1}{2}p\Delta x^2 - q\Delta x\Delta y$

$c_4: -\int_0^{\Delta y} r\cdot 0 + st \, dt = -\frac{1}{2}\left[\frac{1}{2}st^2\right]_0^{\Delta y} = -\frac{1}{2}s\Delta y^2$

$$\oint_{\partial\Delta D} \underline{F}\cdot\underline{ds} = r\Delta x\Delta y - q\Delta x\Delta y = (r-q)\Delta A = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\Delta A$$

$$\oint_D \underline{F}\cdot ds = \sum_{ij}(r-q)\Delta A_{ij} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \quad \square.$$

<u>Example</u> $\oint_C \langle xy^2, x \rangle \cdot \underline{ds}$ $\quad C =$ unit circle oriented counterclockwise



line integral $c(\theta) = (\cos\theta, \sin\theta) \quad 0 \leq \theta \leq 2\pi$

$c'(\theta) = (-\sin\theta, \cos\theta)$

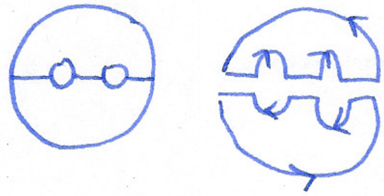
$$\int_0^{2\pi} \langle \cos\theta\sin^2\theta, \cos\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \, d\theta = \int_0^{2\pi} -\cos\theta\sin^3\theta + \cos^2\theta \, d\theta$$

$$= \left[-\frac{1}{4}\sin^4\theta + \frac{1}{2}\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \pi.$$

<u>double integral</u> $\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \iint_D 1 - \underbrace{2xy}_{0 \text{ by symmetry}} dA = \iint_D 1 \, dA = \text{area} = \pi.$

multiple boundary components : orient components

think :

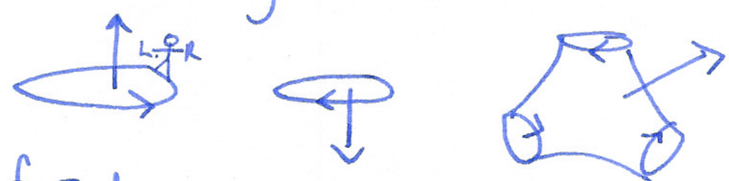


§17.2 Stoke's Theorem (3d version of Green's Thm)

Thm $\oint_{\partial S} \underline{F} . d\underline{s} = \iint_S \underbrace{\nabla \times \underline{F}}_{\text{curl}(\underline{F})} . dS$

S surface (oriented)

∂S boundary (induced orientation)



Remark if S is closed ($\partial S = \phi$) then $\oint_{\phi} \underline{F} . d\underline{s} = 0$

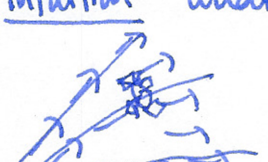
$$\text{curl}(\underline{F}) = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\underset{y}{\partial x_2}} - \frac{\partial F_2}{\underset{z}{\partial x_3}}, -\frac{\partial F_3}{\underset{z}{\partial x_1}} + \frac{\partial F_1}{\underset{z}{\partial x_3}}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right\rangle$$

↑ vector! Green's Thm !

Example $\underline{F} = \underline{x} = \langle x, y, z \rangle$ $\quad \nabla \times \underline{F} = \underline{0}$

$$\underline{F} = \langle xy, e^x, \frac{y}{z} \rangle \qquad \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & e^x & \frac{y}{z} \end{vmatrix} = \langle \frac{1}{z} - 0, -0+0, e^x - x \rangle$$

intuition what does curl($\underline{F}$) measure?



fluid flow : imagine flowing along the vector field

curl($\underline{F}$) = ($\frac{1}{2}$) angular velocity of a small paddle wheel.

Example $\underline{F} = \langle y, 0, 0 \rangle$ $\quad \nabla \times \underline{F} = \langle 0, 0, -1 \rangle$