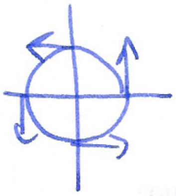
② integrate around unit circle $c(\theta) = \langle \cos\theta, \sin\theta \rangle \quad 0 \le \theta \le 2\pi$

$c'(\theta) = \langle -\sin\theta, \cos\theta \rangle$



$$\int_c \underline{F} \cdot \underline{ds}$$

$$= \int_0^{2\pi} \underline{F}(\underline{c}(\theta)) \cdot \underline{c}'(\theta)\, d\theta$$

$$= \int_0^{2\pi} \left\langle \frac{-\sin\theta}{1}, \frac{\cos\theta}{1} \right\rangle \cdot \langle -\sin\theta, \cos\theta \rangle d\theta = \int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta = \int_0^{2\pi} 1 \, d\theta$$

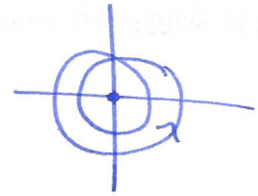
$$= [\theta]_0^{2\pi} = 2\pi \neq 0!$$

③ find potential function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ ($= \theta$ in polars!)

check: $\nabla f = \left\langle \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2}, \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{1}{x} \right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

④ $\tan^{-1}\left(\frac{y}{x}\right)$ not defined at $(0,0)$ so $D = \mathbb{R}^2 \setminus (0,0)$ not simply connected.

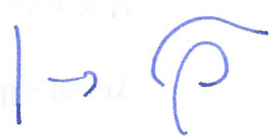
⑤ fact: $\oint_C \underline{F} \cdot \underline{ds} = 2\pi n \quad n =$ winding number



§16.4 Parameterized surfaces and surface integrals

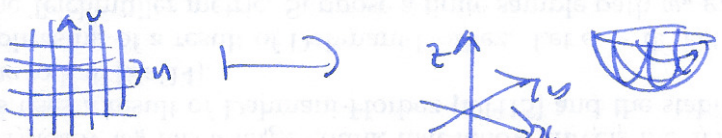
recall. parameterized curve: $\underline{c}(t) : \mathbb{R} \to \mathbb{R}^3$

$t \mapsto (x(t), y(t), z(t))$



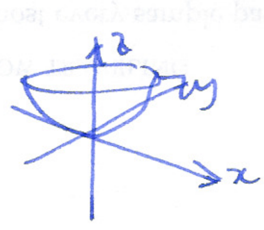
parameterized surface: $\phi \; \mathbb{R}^2 \to \mathbb{R}^3$

$(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$



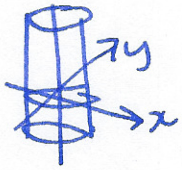
Examples ① paraboloid $z = x^2 + y^2$

$\phi(u,v) = (u, v, u^2+v^2)$.



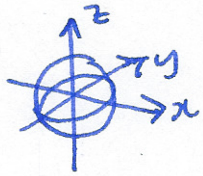
<u>Note</u>: we can parameterize any graph $z=f(x,y)$ by $(u,v)\mapsto(u,v,f(u,v))$

② cylinder $x^2+y^2=1$



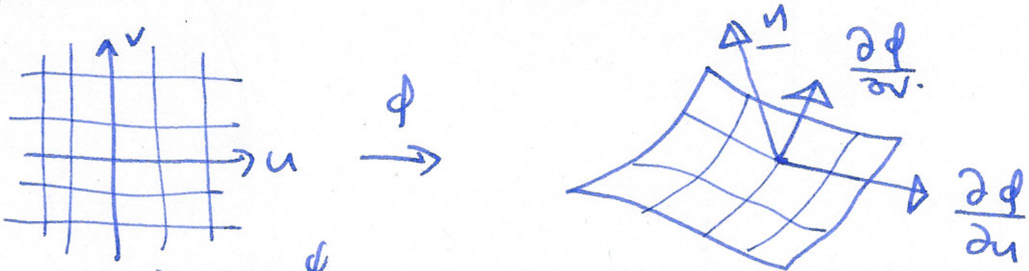
$(\theta,z)\mapsto(\cos\theta,\sin\theta,z)$
$0\leq\theta\leq 2\pi$

③ sphere
$x^2+y^2+z^2=1$



spherical coords, $\rho=1$. $(\theta,\varphi)\mapsto(\cos\theta\sin\varphi,\sin\theta\sin\varphi,\cos\varphi)$
$0\leq\theta\leq2\pi$ $\quad 0\leq\varphi\leq\pi$.

coordinate lines



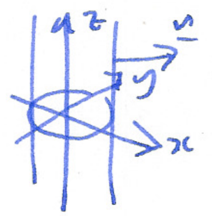
$(u,v)\overset{\varphi}{\longmapsto}(x(u,v),y(u,v),z(u,v))$

$\frac{\partial\varphi}{\partial u}=\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u}\right)=$ tangent vector to surface in u-direction

$\frac{\partial\varphi}{\partial v}=\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v}\right)=$ tangent vector to surface in v-direction.

<u>Q</u>: how do we find the normal vector to the surface?

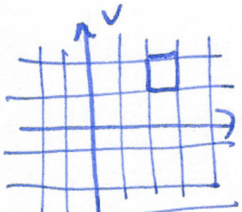
<u>A</u>: $\underline{n}=\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}$ is the normal vector.



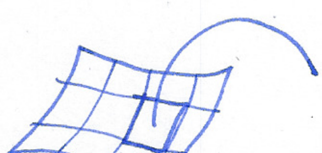
<u>Example</u> cylinder $(\theta,z)\xrightarrow{\varphi}(\cos\theta,\sin\theta,z)$

$\frac{\partial\varphi}{\partial\theta}=(-\sin\theta,\cos\theta,0)$ $\quad\frac{\partial\varphi}{\partial z}=(0,0,1)$ $\quad\underline{n}=\begin{vmatrix} i & j & k\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\langle\cos\theta,\sin\theta,0\rangle$

<u>surface area</u> $\quad\varphi:\mathbb{R}^2\to\mathbb{R}^3$



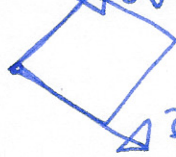
$\xrightarrow{\varphi}$



need scaling function for this piece

$\Delta A=\Delta u\Delta v$

linear approx
$\varphi(u_0,v_0)$



area of parallelogram is $\left\|\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v}\right\|=\|\underline{n}(u_0,v_0)\|$

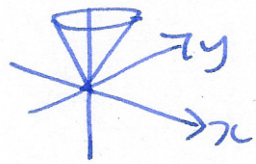
so surface area: $\quad\text{area}(S)=\iint_R\|\underline{n}(u,v)\|\,du\,dv$

How to integrate a scalar function $f(x,y,z)$ over S:

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,z)\,\|\underline{n}(u,v)\|\,du\,dv$$

R ↖ S with same choice of parameterization $\phi(u,v)$.

<u>Example</u> find surface area of cone



$0 \leq z \leq 2$

$\phi(\theta,t) = (t\cos\theta, t\sin\theta, t)$

$0 \leq \theta \leq 2\pi \qquad 0 \leq t \leq 2$

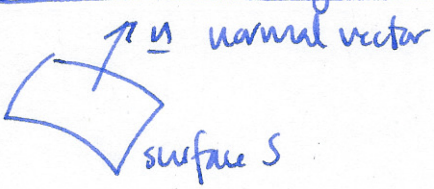
$\frac{\partial\phi}{\partial t} = (\cos\theta, \sin\theta, 1) \quad \frac{\partial\phi}{\partial\theta} = (-t\sin\theta, t\cos\theta, 0)$

$$\underline{n} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -t\sin\theta & t\cos\theta & 0 \end{vmatrix} = \langle -t\cos\theta, -t\sin\theta, t\rangle \qquad \|\underline{n}\| = \sqrt{2t^2} = \sqrt{2}\,t$$

area: $\displaystyle\int_0^{2\pi}\int_0^2 1\cdot\sqrt{2}\,t\,dt\,d\theta = \left[\frac{\sqrt{2}\,t^2}{2}\right]_0^2 = 2\sqrt{2}$

$\displaystyle\int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\sqrt{2}\,\pi.$

<u>§16.5 Vector integrals over surfaces</u>



integrate a vector field $\underline{F}$ over S

notation: $\displaystyle\iint_S \underline{F}\cdot d\underline{S} = \iint_S (\underline{F}\cdot\hat{\underline{n}})\,dS$

$\hat{n}$ unit normal.

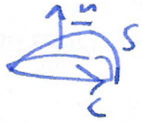
this is called the flux of $\underline{F}$ across S

in terms of a parameterization for S: $\phi(u,v)$ $\quad\displaystyle\iint_D \underline{F}(\phi(u,v))\cdot\underline{n}(u,v)\,du\,dv.$

<u>Example</u> $\underline{F}$ = fluid flow

$\iint \underline{F}\cdot d\underline{S}$ = amount of fluid flowing through surface

<u>Electricity</u> $\underline{E}$ electric field
$\underline{B}$ magnetic field.



Faraday's Law of induction:

$$\int_C \underline{E}\cdot d\underline{s} = -\frac{d}{dt}\iint_S \underline{B}\cdot d\underline{S}.$$