$$= \int_0^{2\pi} -18\sin\theta - 12\sin^2\theta - 6\cos\theta \, d\theta = -12\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

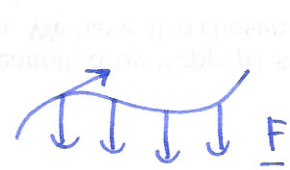
$$= \int_0^{2\pi} -6 + 6\cos 2\theta \, d\theta = -6\int_0^{2\pi} d\theta = -12\pi.$$

useful properties $\int_C (\underline{F} + \underline{G}) \cdot d\underline{s} = \int_C \underline{F} \cdot d\underline{s} + \int_C \underline{G} \cdot d\underline{s}$

$$\int_C k\underline{F} \cdot d\underline{s} = k \int \underline{F} \cdot d\underline{s}$$

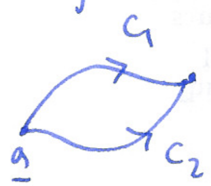
reverse orientation on C : $\int_C \underline{F} \cdot d\underline{s} = -\int_{-C} \underline{F} \cdot d\underline{s}$

Physical interpretation work done $W = \int_C \underline{F} \cdot d\underline{s}$



§16.3 Conservative vector fields

In general $\int_C \underline{F} \cdot d\underline{s}$ depends on the path C, not just the endpoints.



$$\int_{C_1} \underline{F} \cdot d\underline{s} \neq \int_{C_2} \underline{F} \cdot d\underline{s}$$

However for special vector fields $\underline{F}$, $\int_C \underline{F} \cdot d\underline{s}$ only depends on the endpoints. These vector fields are called conservative, i.e. if C_1 and C_2 are two paths with the same endpoints,

$$\underline{F} \text{ conservative} \Rightarrow \int_{C_1} \underline{F} \cdot d\underline{s} = \int_{C_2} \underline{F} \cdot d\underline{s}$$

special case C closed curve, $\underline{F}$ conservative $\Rightarrow \int_C \underline{F} \cdot d\underline{s} = 0$

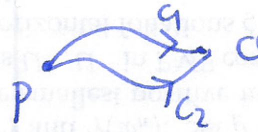
notation : sometimes write $\oint_C \underline{F} \cdot d\underline{s}$.

recall : if $\underline{F}$ is a gradient vector field, then $\underline{F} = \nabla f$ for some scalar function f

<u>Thm</u> (Fundamental theorem for gradient vector fields)

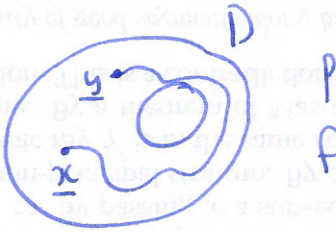
If $\underline{F} = \nabla f$ on a domain D, then for every oriented curve C in D with initial point P and final point Q $\int_C \underline{F} \cdot \underline{ds} = f(Q) - f(P)$

(if C is closed $P = Q$ and so $\oint_C \underline{F} \cdot \underline{ds} = 0$)



<u>Thm</u> Every conservative vector field $\underline{F}$ on an (open connected) domain D is a gradient vector field, i.e. $\underline{F} = \nabla f$ for some f.

<u>Proof</u> (sketch)



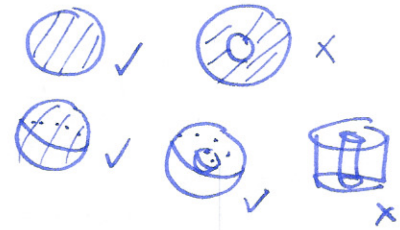
pick a point $\underline{x} \in D$ and define

$f(\underline{y}) = \int_C \underline{F} \cdot \underline{ds}$ where C is any path from $\underline{x}$ to $\underline{y}$. <u>Note</u>: this is well defined if $\underline{F}$ is conservative! Now compute $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ by choosing short horizontal, vertical paths. $\square$.

<u>Q</u>: when is $\underline{F}$ conservative? (i.e. has a potential function).

<u>Thm</u> Let $\underline{F} = \langle F_1, F_2, F_3 \rangle$ be a vector field on a <u>simply connected</u> domain D. Then if the mixed partials are equal: $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$, then $\underline{F} = \nabla f$ for some f.

simply connected : every loop can be shrunk to a point



<u>Example</u> vortex vector field $\underline{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

① mixed partials equal:

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) \cdot 1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1) - (-y) \cdot 2y}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

equal!