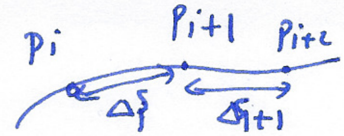
§16.2 Line integrals

parameterized curve C $\underline{c}(t)$



scalar line integral: $\int_C f(x,y,z)\,ds$ f scalar function $f:\mathbb{R}^3 \to \mathbb{R}$

can define a Riemann sum $\int_C f(x,y,z)\,ds = \lim_{\|\Delta s_i\| \to 0} \sum_{i=1}^{n} f(p_i)\,\Delta s_i$

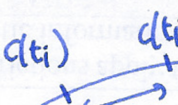
Thm $\underline{c}(t)$ $a \le t \le b$ parameterized curve c.

then $\int_C f(x,y,z)\,dz = \int_a^b f(\underline{c}(t))\,\|\underline{c}'(t)\|\,dt$

this does not depend on the choice of parameterization!

Note: if $f(x,y,z) = 1$, then get $\int_a^b \|\underline{c}(t)\|\,dt$ = arc length of curve.

Intuition:



length of this segment is $\int_{t_i}^{t_{i+1}} \|c'(t)\|\,dt$

so want $\lim_{n\to\infty} \sum_i f(t_i) \int_{t_i}^{t_{i+1}} \|\underline{c}'(t)\|\,dt \sim \int_a^b f(t)\,\|\underline{c}'(t)\|\,dt$

Notation: $ds = \|\underline{c}'(t)\|\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

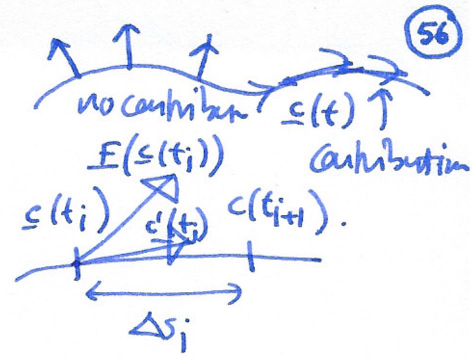
Example find $\int_C (x+y+z)\,ds$ where C is the helix

$\underline{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$.

$\underline{c}'(t) = (-\sin t, \cos t, 1)$

$\int_0^{2\pi} (\cos t + \sin t + t)\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = 2\int_0^{2\pi} \cos t + \sin t + t\,dt$

$= 2\left[\sin t - \cos t + \frac{1}{2}t^2\right]_0^{2\pi} = \frac{1}{2}(2\pi)^2 = 2\pi^2$



Vector line integrals notation $\int_C \underline{F}.\underline{ds}$

intuition: integral of tangential component of $\underline{F}$ along c

Riemann sum def$^{\underline{n}}$: $\sum_i F(\underline{c}(t_i)).\underline{c}'(t_i)\Delta s_i$

Def$^{\underline{n}}$ let $T = \dfrac{\underline{c}'(t)}{\|\underline{c}'(t)\|}$ be the unit tangent vector to $\underline{c}$

then $\int_C \underline{F}.\underline{ds} = \int_C (\underline{F}.\underline{T})\,ds$

If $\underline{c}$ has parameterization $\underline{c}(t)$ for $a \le t \le b$, then

$$\int_C \underline{F}.\underline{ds} = \int_a^b \underline{F}(\underline{c}(t)).\frac{\underline{c}'(t)}{\|\underline{c}(t)\|}\|\underline{c}'(t)\|\,dt = \int_a^b \underline{F}(\underline{c}(t)).\underline{c}'(t)\,dt$$

summary $\boxed{\int_C \underline{F}.\underline{ds} = \int_a^b \underline{F}(\underline{c}(t))\cancel{\theta}.\underline{c}'(t)\,dt}$

Fact: this does not depend on the parameterization.

Alternate notation: $\int_C \underline{F}.\underline{ds}$ $\quad \underline{F} = \langle F_1, F_2, F_3 \rangle$

then write $\int_C \underline{F}.\underline{ds} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$

in terms of $\underline{c}(t)$: $= \int_a^b F_1(\underline{c}(t))\frac{dx}{dt}dt + F_2(\underline{c}(t))\frac{dy}{dt}dt + F_3(\underline{c}(t))\frac{dz}{dt}dt$

(same as before).

Example (2d) integrate $\underline{F} = \langle 2y, -3 \rangle$ over the ellipse $\underline{c}(t) = (4+3\cos\theta, 3+2\sin\theta)$
$0 \le \theta \le 2\pi$

$\underline{c}'(t) = \langle -3\sin\theta, 2\cos\theta \rangle$

$$\int_0^{2\pi} \underline{F}.\underline{c}'(\theta)\,d\theta = \int_0^{2\pi} \overset{6+4\sin\theta,\ -3}{\langle \cancel{-3\sin\theta, 2\cos\theta} \rangle}.\langle -3\sin\theta, 2\cos\theta \rangle\,d\theta$$