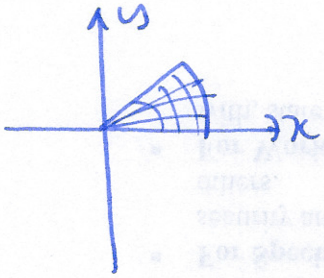
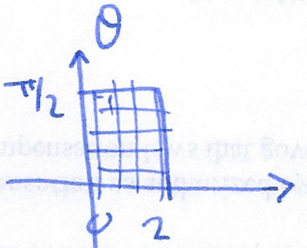
Example polar coordinates

$$\Phi(r,\theta) = (\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})$$





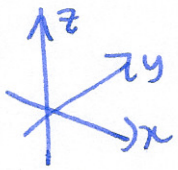
find $J(\Phi)$:

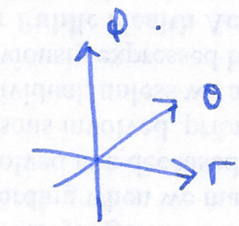
$$J(\Phi) = \left|\begin{matrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{matrix}\right| = \left|\begin{matrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{matrix}\right| = r\cos^2\theta + r\sin^2\theta = r$$

so $dx\,dy = r\,dr\,d\theta$

useful fact $J(\Phi) = \dfrac{1}{J(\Phi^{-1})}$ $\quad \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|}$.

Example spherical coordinates





$$dx\,dy\,dz = J\,d\rho\,d\theta\,d\phi.$$

$$J = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \left|\begin{matrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{matrix}\right|$$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$$J = \left|\begin{matrix} \cos\theta\sin\phi & -\rho\sin\phi\sin\theta & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{matrix}\right|$$

$$= \cos\phi\left|\begin{matrix} -\rho\sin\phi\sin\theta & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \end{matrix}\right| - \rho\sin\phi\left|\begin{matrix} \cos\theta\sin\phi & -\rho\sin\phi\sin\theta \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{matrix}\right|$$

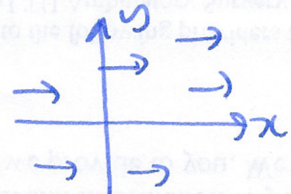
$$= \rho^2\Big[\cos\phi\underbrace{\left(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi\right)}_{-\sin\phi\cos\phi(\sin^2\theta+\cos^2\theta)} - \sin^3\phi\underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{1}\Big].$$

$$= \rho^2\sin\phi\left(\sin^2\phi + \cos^2\phi\right) = \rho^2\sin\phi.$$

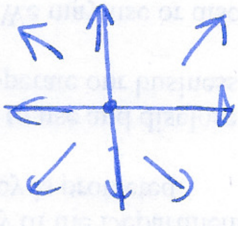
§ 16.1 Vector fields

Defn: A vector field is a function which assigns a vector to each point in the domain.

Example (2d) ① $F(x,y) = \langle 1, 0 \rangle$



② $F(x,y) = \langle x, y \rangle$



③ $F(x,y) = \langle -y, x \rangle$

④ any gradient vector field $F(x,y) = \nabla f \qquad f: \mathbb{R}^2 \to \mathbb{R}$.

Defn: A vector field is conservative if $F = \nabla f$ for some scalar function f, called the potential function for F.

Example (3d) (inverse square field) $F(x,y,z) = F(\underline{x}) = \frac{1}{\|\underline{x}\|^3}\underline{x}$

(note: $\|F(\underline{x})\| = \frac{1}{\|\underline{x}\|^2}$

claim: this is a conservative vector field, with potential function

$$f(\underline{x}) = -\frac{1}{\|\underline{x}\|} = \frac{-1}{\sqrt{x^2+y^2+z^2}} = -(x^2+y^2+z^2)^{-1/2}$$

check: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle \frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x, \ \ldots \ \rangle$

$$= \frac{\langle x, y, z \rangle}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\underline{x}}{\|\underline{x}\|^3}.$$

Warning: not all vector fields are conservative

Observation: if $F = \nabla f$, then mixed partials are equal.

example (2d) $F(x,y) = \langle y, x \rangle \qquad \frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}(x)$ mixed partials are equal ✓

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = y \Rightarrow f(x,y) = xy + c_1(y) \\ \frac{\partial f}{\partial y} = x \Rightarrow f(x,y) = xy + c_2(x) \end{array}\right\} \text{ but } \left.\begin{array}{l} \frac{\partial}{\partial y} c_1 = 0 \\ \frac{\partial c_2}{\partial x} = 0 \end{array}\right\} \Rightarrow f(x,y) = xy + c$$

c constant.