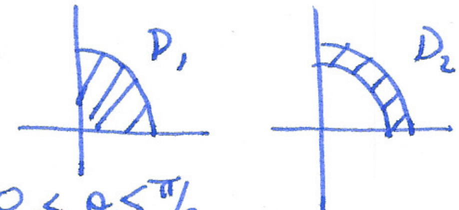
§15.4 Integration in polar, cylindrical and spherical coordinates

recall: double integrals over regions $\iint_D f(x,y)\,dA$

some regions easier to describe in polars.



$0 \le \theta \le \pi/2$, $0 \le r \le 1$ $\qquad$ $0 \le \theta \le \pi/2$, $1 \le r \le 2$

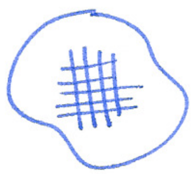
useful fact: in polar coords $\iint_D f(x,y)\,dA = \iint_D f(r,\theta)\,r\,dr\,d\theta$

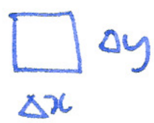
$$f(r,\theta) = f(x(r,\theta), y(r,\theta)).$$

i.e. $dA = dx\,dy$ in cartesians

$\boxed{dA = r\,dr\,d\theta}$ in polars $\qquad$ warning: do not forget the r

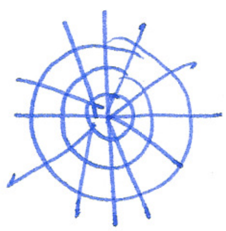
idea: cartesians:





area $\Delta A = \Delta x \Delta y$

polars:



"squares" are different sizes

area $\Delta A \approx r \Delta r \Delta \theta$

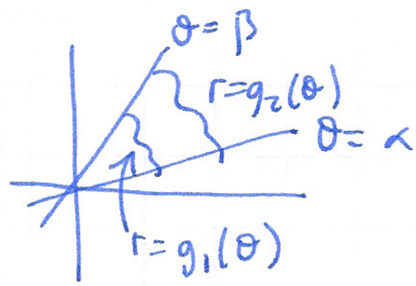
area of wedge

area is

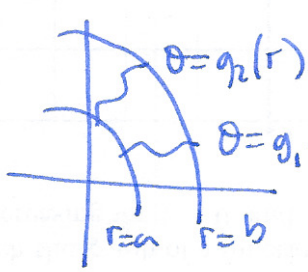
$$(r+\Delta r)\Delta\theta - r\Delta\theta \approx r \Delta r \Delta\theta$$

Describing regions

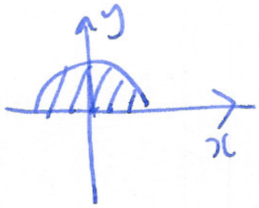
$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$



$$\int_a^b \int_{g_1(r)}^{g_2(r)} f(r,\theta)\,r\,d\theta\,dr$$



Example Integrate $f(x,y) = xy$ over the upper half disc of radius 4



$0 \leq \theta \leq \pi$
$1 \leq r \leq 4$

$$\iint_D f(x,y)\,dA = \int_0^{\pi}\int_0^{4} r^2\cos\theta\sin\theta \, r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\sin 2\theta \, \cancel{d\theta} \underbrace{\int_0^4 r^3\,dr}_{\left[\frac{1}{4}r^4\right]_0^4} \, d\theta = 32\left[-\frac{1}{4}\cos 2\theta\right]_0^{\pi} = 0.$$

Cylindrical coords $\iiint_R f(x,y,z)\,dV$ $\boxed{dV = r\,dr\,d\theta\,dz}$

spherical coords $\boxed{dV = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}$

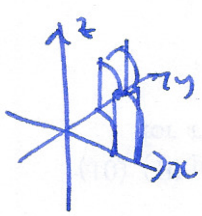
Example $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta \qquad \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \frac{1}{2}$$

$$\int_0^{2\pi} \frac{1}{2}\,d\theta = \pi.$$

Example



$$\iiint_R \frac{1}{x^2+y^2\cancel{+z^2}}\,dV \qquad \int_0^1\int_0^{\pi/2}\int_2^1 \frac{1}{r^2}\, r\,dr\,d\theta\,dz$$

$1 \leq r \leq 2$
$0 \leq \theta \leq \pi/2$
$0 \leq z \leq 1$

$$= \int_1^2 \frac{1}{r}\,dr = \left[\ln|r|\right]_1^2 = \ln(2)$$

$$\int_0^{\pi/2} \ln(2)\,d\theta = \left[\theta\ln(2)\right]_0^{\pi/2} = \frac{\pi}{2}\ln(2).$$

$$\int_0^1 \frac{\pi}{2}\ln(2)\,dz = \frac{\pi}{2}\ln(2).$$

§15.5 Change of variable

1d $\displaystyle\int_a^b f(x)\,dx \qquad x(u) \longleftrightarrow \int_c^d f(x(u))\,\frac{dx}{du}\,du \qquad \begin{array}{l} x(d)=b \\ x(c)=a \end{array}$

2d $\displaystyle\iint_D f(x,y)\,dA \qquad \begin{array}{l} x(u,v) \\ y(u,v)\end{array} \Longleftrightarrow \iint f(x(u,v),y(u,v))\,J\,du\,dv$

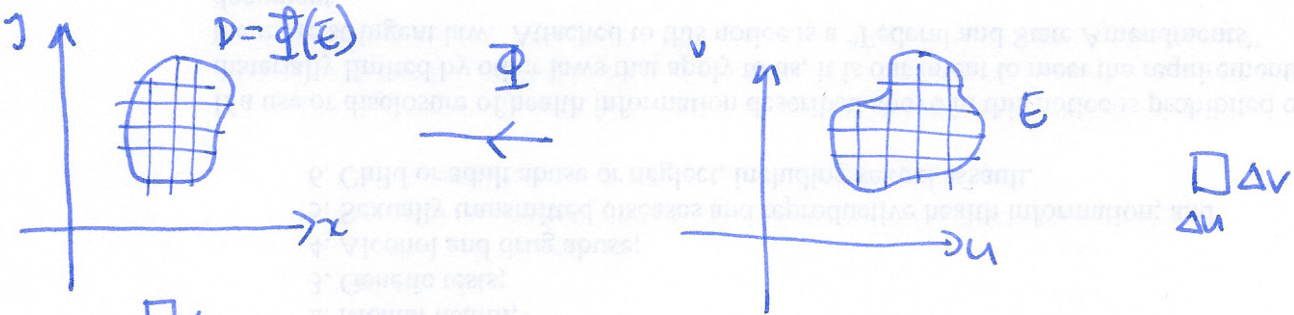
$dA = dx\,dy$ or $dy\,dx$

limits describe D in terms of u,v

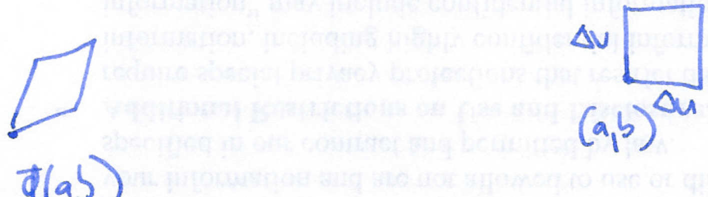
in terms of u,v

$J = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right|$

$D = \Phi(E)$



$\square\,\Delta y$, Δx $\qquad (x(u,v),y(u,v)) \longleftarrow (u,v)$



linear approximation: $(u,v) \overset{\Phi}{\longmapsto} (x(u,v),y(u,v))$

$$\approx \Big(x(a,b) + \frac{\partial x}{\partial u}(a,b)(u-a) + \frac{\partial x}{\partial v}(a,b)(v-b),\; y(a,b) + \frac{\partial y}{\partial u}(a,b)(u-a) + \frac{\partial y}{\partial v}(a,b)(v-b)\Big)$$

$$= (x(a,b),y(a,b)) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\begin{bmatrix} u-a \\ v-b\end{bmatrix}$$

need to replace $\Delta x \Delta y$ with $\left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right| \Delta u \Delta v$ or $dx\,dy = J(\Phi)\,du\,dv$